

Find the root of the equation given below by bisection method.

$$x^3 - x^2 + x - 7 = 0$$

(Note: accuracy up to three decimal places is required.)

Marks: 10

**SOLUTION**

**Let**

$$f(x) = x^3 - x^2 + x - 7$$

$$\text{Now } f(1) = (1)^3 - (1)^2 + 1 - 7 = -6 < 0$$

$$f(2) = (2)^3 - (2)^2 + 2 - 7 = -1 < 0$$

$$\text{and } f(3) = (3)^3 - (3)^2 + 3 - 7 = 14 > 0$$

$$\text{As } f(2)f(3) = -14 < 0$$

Therefore a real root lies between 2 and 3

**Iteration 1**

Let

$$x_0 = 2 \text{ and } x_1 = 3$$

then

$$x_2 = \left(\frac{x_0 + x_1}{2}\right)$$

$$= \left(\frac{2+3}{2}\right) = 2.5$$

$$\text{Now } f(2.5) = (2.5)^3 - (2.5)^2 + 2.5 - 7 = 4.875 > 0$$

$$\text{As } f(2)f(2.5) = -1(4.875) = -4.875 < 0$$

Therefore a real root lies between 2 and 2.5

**Iteration 2**

$$x_0 = 2 \text{ and } x_2 = 2.5$$

then

$$x_3 = \left(\frac{x_0 + x_2}{2}\right)$$

$$= \left(\frac{2+2.5}{2}\right) = 2.25$$

$$\text{Now } f(2.25) = (2.25)^3 - (2.25)^2 + 2.25 - 7 = 1.578125 > 0$$

$$\text{As } f(2)f(2.25) = -1(1.578125) = -1.578125 < 0$$

Therefore a real root lies between 2 and 2.25

**Iteration 3**

$$x_0 = 2 \text{ and } x_3 = 2.25$$

then

$$\begin{aligned}x_4 &= \left(\frac{x_0 + x_3}{2}\right) \\ &= \left(\frac{2 + 2.25}{2}\right) = 2.125\end{aligned}$$

$$\text{Now } f(2.125) = (2.125)^3 - (2.125)^2 + 2.125 - 7 = 0.205078 > 0$$

$$\text{As } f(2)f(2.125) = -1(0.205078) = -0.205078 < 0$$

Therefore a real root lies between 2 and 2.125

#### Iteration 4

$$x_0 = 2 \text{ and } x_4 = 2.125$$

then

$$\begin{aligned}x_5 &= \left(\frac{x_0 + x_4}{2}\right) \\ &= \left(\frac{2 + 2.125}{2}\right) = 2.0625\end{aligned}$$

$$\text{Now } f(2.0625) = (2.0625)^3 - (2.0625)^2 + 2.0625 - 7 = -0.417725 < 0$$

$$\text{As } f(2.0625)f(2.125) = -0.417725(0.205078) = -0.085666 < 0$$

Therefore a real root lies between 2.0625 and 2.125

#### Iteration 5

$$x_5 = 2.0625 \text{ and } x_4 = 2.125$$

then

$$\begin{aligned}x_6 &= \left(\frac{x_5 + x_4}{2}\right) \\ &= \left(\frac{2.0625 + 2.125}{2}\right) = 2.09375\end{aligned}$$

$$\text{Now } f(2.09375) = (2.09375)^3 - (2.09375)^2 + 2.09375 - 7 = -0.111481 < 0$$

$$\text{As } f(2.09375)f(2.125) = -0.111481(0.205078) = -0.022862 < 0$$

Therefore a real root lies between 2.09375 and 2.125

#### Iteration 6

$$x_6 = 2.09375 \text{ and } x_4 = 2.125$$

then

$$\begin{aligned}x_7 &= \left(\frac{x_6 + x_4}{2}\right) \\ &= \left(\frac{2.09375 + 2.125}{2}\right) = 2.109375\end{aligned}$$

$$\text{Now } f(2.109375) = (2.109375)^3 - (2.109375)^2 + 2.109375 - 7 = 0.0455 > 0$$

$$\text{As } f(2.09375)f(2.109375) = -0.111481(0.0455) = -0.00507 < 0$$

Therefore a real root lies between 2.09375 and 2.109375

#### Iteration 7

$$x_6 = 2.09375 \text{ and } x_7 = 2.109375$$

then

$$\begin{aligned}x_8 &= \left(\frac{x_6 + x_7}{2}\right) \\ &= \left(\frac{2.09375 + 2.109375}{2}\right) = 2.1015625\end{aligned}$$

$$\text{Now } f(2.1015625) = (2.1015625)^3 - (2.1015625)^2 + 2.1015625 - 7 = -0.0333 < 0$$

$$\text{As } f(2.1015625)f(2.109375) = -0.0333(0.0455) = -0.01515 < 0$$

Therefore a real root lies between 2.1015625 and 2.109375

#### Iteration 8

$$x_8 = 2.1015625 \text{ and } x_7 = 2.109375$$

then

$$\begin{aligned}x_9 &= \left(\frac{x_8 + x_7}{2}\right) \\ &= \left(\frac{2.1015625 + 2.109375}{2}\right) = 2.10546875\end{aligned}$$

$$\text{Now } f(2.10546875) = (2.10546875)^3 - (2.10546875)^2 + 2.10546875 - 7 = 0.006 > 0$$

$$\text{As } f(2.1015625)f(2.10546875) = -0.0333(0.006) = -0.0002 < 0$$

Therefore a real root lies between 2.1015625 and 2.10546875

#### Iteration 9

$$x_8 = 2.1015625 \text{ and } x_9 = 2.10546875$$

then

$$\begin{aligned}x_{10} &= \left(\frac{x_8 + x_9}{2}\right) \\ &= \left(\frac{2.1015625 + 2.10546875}{2}\right) = 2.103515625\end{aligned}$$

$$\text{Now } f(2.103515625) = (2.103515625)^3 - (2.103515625)^2 + 2.103515625 - 7 = -0.01367 < 0$$

$$\text{As } f(2.103515625)f(2.10546875) = -0.01367(0.006) = -0.00008 < 0$$

Therefore a real root lies between 2.103515625 and 2.10546875

#### Iteration 10

$$x_{10} = 2.103515625 \text{ and } x_9 = 2.10546875$$

then

$$\begin{aligned}x_{11} &= \left(\frac{x_{10} + x_9}{2}\right) \\ &= \left(\frac{2.103515625 + 2.10546875}{2}\right) = 2.104492188\end{aligned}$$

$$\text{Now } f(2.104492188) = (2.104492188)^3 - (2.104492188)^2 + 2.104492188 - 7 = -0.0038 < 0$$

$$\text{As } f(2.104492188)f(2.10546875) = -0.0038(0.006) = -0.0000228 < 0$$

Therefore a real root lies between 2.104492188 and 2.10546875

#### Iteration 11

$$x_{11} = 2.104492188 \text{ and } x_0 = 2.10546875$$

then

$$x_{12} = \frac{(x_{11} + x_0)}{2} \\ = \frac{(2.104492188 + 2.10546875)}{2} = 2.104980469$$

$$\text{Now } f(2.104980469) = (2.104980469)^3 - (2.104980469)^2 + 2.104980469 - 7 = 0.0011 > 0$$

$$\text{As } f(2.104492188)f(2.104980469) = -0.0038(0.0011) = -0.00000418 < 0$$

Therefore a real root lies between 2.104492188 and 2.104980469

As the next root lies between 2.104492188 and 2.104980469 and these roots are equal up to three decimal places.

So, the required root up to three decimal places is **2.104**

**Example From Handout at page # 4**

Solve  $x^3 - 9x + 1 = 0$  for the root between  $x=2$  and  $x=4$  by bisection method

Solution:

Here we are given the interval (2,4) so we need not to carry out intermediate value property to locate initial approximation.

$$\text{Here } f(x) = x^3 - 9x + 1 = 0$$

$$\text{Now } f(2) = -9 \text{ and } f(4) = 29.$$

Here  $f(2).f(4) < 0$  So the root lies between 2 and 4.

$$\text{So, } x_0 = 2, x_1 = 4$$

$$\text{and } x_2 = \frac{x_0 + x_1}{2} = 3 \text{ (This formula predicts next iteration)}$$

Now  $f(3) = 1$ , Here  $f(2).f(3) < 0$  So the root lies between 2 and 3.

Repeat above process to required number of iteration.

Step, Take two Initial Approximation such that  $f(x_1).f(x_2) < 0$ . Means both must have opposite signs. Take their mean by  $x_3 = (x_1 + x_2)/2$

Next Take two element from which 1 will be  $x_3$  and another from  $x_1$  or  $x_2$  such that both  $x_3$  and the other element should have opposite sign.

Repeat the above process to the required numbers of iterations.

### Regula Falsi Method

Formula for the Regula Falsi Method is  $x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$  or

$$x_{(n+1)} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

Steps: If the interval is given, check whether the root lies in between the interval or not by the principle that states if both number have opposite sign, then the root lies in between interval.

Find the next approximation with the help of the formula.

**Question#2**

**Marks 10**

Use Regula-Fasli method to compute the root of the equation  $f(x) = \cos x - xe^x$  In the interval  $[0, 1]$  after third iteration.

**Solution:**

$$\text{As } f(x) = \cos x - xe^x$$

$$f(0) = \cos 0 - 0e^0 = 1 > 0$$

$$f(1) = \cos 1 - 1e^1 = 0.5403 - 2.718 = -2.1779 < 0$$

So root of the eq. lie between 0 and 1. Let  $x_0 = 0$  and  $x_1 = 1$

$$\text{Thus } f(x_0) = 1 \text{ and } f(x_1) = -2.1779$$

The formula for finding the root of the function  $f(x)$  by Regula-Fasli is given by.

$$x_{(n+1)} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

For  $n=1$  we have,

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 1 - \frac{1-0}{-2.1779-1} (-2.1779) = 1 - \frac{2.1779}{3.1779} = 1 - 0.68533 = 0.31467$$

$$f(x_2) = \cos(0.31467) - (0.31467)e^{0.31467} = 0.950899 - (0.31467)(1.369806) = 0.519863$$

Since  $f(x_1)$  and  $f(x_2)$  are of opposite signs, the root lies between  $x_1$  and  $x_2$ .

Therefore, for  $n=2$  we have,

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 0.31467 - \frac{0.31467-1}{0.519863 - (-2.1779)} (0.519863)$$

$$= 0.31467 + \frac{0.68533}{2.697833} (0.519863) = 0.44673$$

$$f(x_3) = \cos(0.44673) - (0.44673)e^{0.44673} = 0.901864 - (0.44673)(1.563191) = 0.20353$$

Since  $f(x_1)$  and  $f(x_3)$  are of opposite signs, the root lies between  $x_1$  and  $x_3$ .

Therefore, for  $n=3$  we have,

$$x_4 = x_3 - \frac{x_3 - x_1}{f(x_3) - f(x_1)} f(x_3) = 0.44673 - \frac{0.44673-1}{0.20353 - (-2.1779)} (0.20353)$$

$$= 0.44673 + \frac{0.54327}{2.3815} (0.20353) = 0.493159$$

The required root after 3<sup>rd</sup> iteration using Regula-Falsi method is 0.493159

**Question 2**

Use the Regula Falsi (method of false position) to solve the equation  $x^3 - 4x - 9 = 0$

**(Note: accuracy up to four decimal places is required)**

**Marks: 10**

**SOLUTION**

Let

$$f(x) = x^3 - 4x - 9$$

Now

$$f(0) = (0)^3 - 4(0) - 9 = 0 - 0 - 9 = -9 < 0$$

$$f(1) = (1)^3 - 4(1) - 9 = 1 - 4 - 9 = -12 < 0$$

$$f(2) = (2)^3 - 4(2) - 9 = 8 - 8 - 9 = -9 < 0$$

$$f(3) = (3)^3 - 4(3) - 9 = 27 - 12 - 9 = 6 > 0$$

Now, Since  $f(2)$  and  $f(3)$  are of opposite signs,  
therefore, the real root lies between 2 and 3

Now,

$$\text{let } x_1 = 2 \text{ and } x_2 = 3$$

### First iteration

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 3 - \frac{3 - 2}{6 - (-9)} (6)$$

$$= 3 - \frac{1}{15} (6)$$

$$= 3 - \frac{2}{5}$$

$$= 2.6$$

$$f(x_3) = f(2.6) = (2.6)^3 - 4(2.6) - 9 \\ = -1.824$$

### Second iteration

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$= 2.6 - \frac{2.6 - 3}{-1.824 - 6} (-1.824)$$

$$= 2.69325$$

$$f(x_4) = f(2.69325) = (2.69325)^3 - 4(2.69325) - 9 \\ = -0.23725$$

### Third iteration

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4)$$

$$= 2.69325 - \frac{2.69325 - 2.6}{-0.23725 - (-1.824)} (-0.2061)$$

$$= 2.70536$$

$$f(x_5) = f(2.70536) = (2.70536)^2 - 4(2.70536) - 9$$

$$= -0.02098$$

#### Fourth iteration

$$x_6 = x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} f(x_5)$$

$$= 2.70536 - \frac{2.70536 - 2.69325}{-0.02098 - (-0.23725)} (-0.02098)$$

$$= 2.70653$$

$$f(x_6) = f(2.70653) = (2.70653)^2 - 4(2.70653) - 9$$

$$= 0.0000368$$

#### Fifth iteration

$$x_7 = x_6 - \frac{x_6 - x_5}{f(x_6) - f(x_5)} f(x_6)$$

$$= 2.70653 - \frac{2.70653 - 2.70536}{0.0000368 - (-0.02098)} (0.0000368)$$

$$= 2.70653$$

Hence,

The required root of the given equation correct to 4 decimal places is **2.7065**

### Method of Iterations

#### Question#1

**Marks 10**

Use the method of iterations to determine the real root of the equation  $e^{-x} = 10x$  in the interval  $[0, 1]$ , correct to four decimal places after four iterations.

#### Solution:

Since  $e^{-x} = 10x$ . Let  $f(x) = e^{-x} - 10x$ . we see that  $f(0) = 1$  and  $f(1) = -9.6321$

As  $e^{-x} = 10x$ , we can easily find the value of  $x$ , thus

$$x = \frac{e^{-x}}{10}, \text{ Let } \phi(x) = \frac{e^{-x}}{10}$$

By taking its derivative we get

$$\phi'(x) = -\frac{e^{-x}}{10}, \text{ we see that } |\phi'(x)| < 1 \text{ for all values in } [0, 1].$$

Therefore we can apply method of iterations to the given function.

Take any value within  $[0,1]$ . Let

$$x_0 = 0.5$$

$$x_1 = \phi(0.5) = -\frac{e^{-0.5}}{10} = 0.0607 \quad , \quad f(x_1) = -0.09516$$

$$x_2 = \phi(0.0607) = -\frac{e^{-0.0607}}{10} = 0.0914 \quad , \quad f(x_2) = 0.00869$$

$$x_3 = \phi(0.0914) = -\frac{e^{-0.0914}}{10} = 0.0941 \quad , \quad f(x_3) = -0.000790477 = -7.90477 \times 10^{-4}$$

$$x_4 = \phi(0.0941) = -\frac{e^{-0.0941}}{10} = 0.0913 \quad , \quad f(x_4) = 0.0000275784 = 2.75784 \times 10^{-5}$$

$$x_4 = 0.0913$$

Example From the Handout at page #

### Example

Find a real root of the equation  $\cos x = 3x - 1$  correct to seven places of decimal.

### Solution

Here it is a transcendental function and all the calculation must be done in the radians mode and value of pi should be 3.14

$$f(x) = \cos x - 3x + 1$$

$$f(0) = \cos 0 - 3(0) + 1 = 1 > 0$$

$$f(\pi/2) = \cos(1.57) - 3(1.57) + 1 = 0.0007963 - 4.71 + 1 = -3.7092037 < 0$$

so a real root lies between 0 and  $\pi/2$

$$\text{here } \phi(x) = \frac{1}{3}(\cos x + 1)$$

$$\text{we have } \phi'(x) = -\frac{1}{3}\sin x$$

it is clearly less than 1 as sin is a bounded function and it's values lies between -1 and 1

hence iteration method can be applied

let  $x_0 = 0.5$  be the initial approximation then

$$x_1 = \phi(x_0) = \frac{1}{3}[\cos(0.5) + 1] = 0.6258608$$

$$x_2 = \phi(x_1) = \frac{1}{3}[\cos(0.6258608) + 1] = 0.6034863$$

$$x_3 = \phi(x_2) = \frac{1}{3}[\cos(0.6034863) + 1] = 0.6077873$$

$$x_4 = \phi(x_3) = \frac{1}{3}[\cos(0.6077873) + 1] = 0.6069711$$

$$x_5 = \phi(x_4) = \frac{1}{3}[\cos(0.6069711) + 1] = 0.6071264$$

$$x_6 = \phi(x_5) = \frac{1}{3}[\cos(0.6071264) + 1] = 0.6070969$$

$$x_7 = \phi(x_6) = \frac{1}{3}[\cos(0.6070969) + 1] = 0.6071025$$

$$x_8 = \phi(x_7) = \frac{1}{3}[\cos(0.6071025) + 1] = 0.6071014$$

$$x_9 = \phi(x_8) = \frac{1}{3}[\cos(0.6071014) + 1] = 0.6071016$$

$$x_{10} = \phi(x_9) = \frac{1}{3}[\cos(0.6071016) + 1] = 0.6071016$$

### Newton-Raphson Method

Procedural Detail:

Find the limit if not provided by starting from  $x=0$  to ,...

Find two consecutive numbers for  $f(x)$  should have opposite sign.

Use the Newton Raphson Formula to find next approximation .

### Question#3

Marks 10

Find the real root of the equation  $x^3 - 3x - 5 = 0$  using Newton-Raphson method in the interval  $[2,3]$  after third iteration.

**Solution:**

$$\text{As } f(x) = x^3 - 3x - 5$$

$$f(2) = 2^3 - 3 \cdot 2 - 5 = -3 < 0$$

$$f(3) = 3^3 - 3 \cdot 3 - 5 = 13 > 0$$

So root of the eq. will lie in  $[2,3]$

$$\text{now } f'(x) = 3x^2 - 3 \text{ and } f''(x) = 6x$$

$$f'(2) = 3 \cdot 2^2 - 3 = 9 \text{ and } f'(3) = 3 \cdot 3^2 - 3 = 24$$

$$f''(2) = 6 \cdot 2 = 12 \text{ and } f''(3) = 6 \cdot 3 = 18$$

Since  $f(3)$  and  $f''(3)$  are of same sign. So we choose  $x_0 = 3$

so by Newton's method we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{13}{24} = 2.4583$$

$$f(x_1) = (2.4583)^3 - 3(2.4583) - 5 = 14.8561 - 7.3749 - 5 = 2.4812$$

$$f'(x_1) = f'(2.4583) = 3(2.4583)^2 - 3 = 18.1297 - 3 = 15.1297$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.4583 - \frac{2.4812}{15.1297} = 2.2943$$

$$f(x_2) = (2.2943)^3 - 3(2.2943) - 5 = 12.0767 - 6.8829 - 5 = 0.1938$$

$$f'(x_2) = f'(2.2943) = 3(2.2943)^2 - 3 = 15.7914 - 3 = 12.7914$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.2943 - \frac{0.1938}{12.7914} = 2.2791$$

Example From the Handout at page #

Value of  $e=2.7182$

$$n = 2$$

$$f(x_2) = f(0.2) = 0.408$$

$$x_3 = \frac{x_1(f(x_2) - x_2 f(x_1))}{(f(x_2) - f(x_1))}$$

$$x_3 = \frac{(0.5)(0.408) - (0.2)(-0.375)}{(0.408) - (-0.375)}$$

$$x_3 = \frac{(0.204) + (0.075)}{(0.408) + (0.375)}$$

$$x_3 = \frac{0.279}{0.783}$$

$$x_3 = 0.3563$$

### ITERATION 3

$$n = 3$$

$$f(x_3) = f(0.3563) = -0.02367$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$x_4 = \frac{(0.2)f(0.3563) - (0.3563)f(0.408)}{f(0.3563) - f(0.408)}$$

$$x_4 = \frac{(0.2)(-0.02367) - (0.3563)(0.408)}{(-0.02367) - (0.408)}$$

$$x_4 = \frac{-0.004737 - 0.1453}{-0.43167}$$

$$x_4 = \frac{-0.150034}{-0.43167}$$

$$x_4 = 0.3477$$

### ITERATION 4

$$n = 4$$

$$f(x_4) = f(0.3477) = -0.02367$$

$$x_4 = 0.3477 \quad f(x_4) = -0.00107$$

$$x_5 = \frac{x_4 f(x_4) - x_3 f(x_3)}{f(x_4) - f(x_3)}$$

$$x_5 = \frac{(0.3563)(-0.00107) - (0.3563)(-0.02367)}{(-0.00107) - (-0.02367)}$$

$$x_5 = \frac{-0.000381 + 0.0823}{-0.00107 + 0.02367}$$

$$x_5 = \frac{0.7849}{0.0226}$$

$$x_5 = 0.3473$$

$$x_5 = 0.3473 \quad f(x_5) = -0.0000096$$

Hence, the root after four iterations is **0.347**

### Question 2

Use the secant method to solve the equation  $e^x - 3x^2$  for  $0 \leq x \leq 1$ . (Perform only 3 iterations.)

### Solution

$$f(x) = e^x - 3x^2, \quad x_0 = 0, \quad x_1 = 1$$

both are the initial approximations

$$f(0) = e^0 - 3(0) = 1$$

$$f(1) = e^1 - 3(1) = -0.281$$

now we calculate the second approximation

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_0) - f(x_1)} = \frac{(0)(-0.281) - (1)1}{-0.281 - 1} = 0.7806$$

$$f(0.7806) = e^{0.7806} - 3(0.7806)^2 = 2.1827 - 1.82800 = 0.3546$$

now  $x_1 = 1$  and  $x_2 = 0.786$

$$f(1) = -0.281 \quad f(0.786) = 0.3546$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{1(0.3546) - (0.786)(-0.281)}{0.3546 + 0.281} = 0.9052$$

now

$$x_3 = 0.9052 \quad f(0.9052) = 2.71^{0.9052} - 3(0.9052)^2 = 0.0074$$

$$x_2 = 0.786 \quad f(0.786) = 0.3546$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{0.7806(0.0074) - (0.9052)(0.3546)}{0.0074 + 0.3546} = 0.9076$$

so root is 0.9076

## Muller's Method

### Question 1

Solve the equation  $x^3 - 7x^2 + 14x - 6$  by using Muller's method only perform three

Iterations. ( $x_0 = 0.5, x_1 = 1, x_2 = 0$ )

### Solution

#### 1<sup>st</sup> Iteration

$x_0 = 0.5, x_1 = 1$  and  $x_2 = 0$

$$f(x_0) = f_0 = f(0.5) = (0.5)^3 - 7(0.5)^2 + 14(0.5) - 6 = -0.625$$

$$f(x_1) = f_1 = f(1) = 1^3 - 7(1)^2 + 14(1) - 6 = 2$$

$$f(x_2) = f_2 = f(0) = (0)^3 - 7(0)^2 + 14(0) - 6 = -6$$

$$c = f_0 = -0.625$$

$$h_1 = x_1 - x_0 = 1 - 0.5 = 0.5$$

$$h_2 = x_0 - x_2 = 0.5 - 0 = 0.5$$

$$a = \frac{h_2 f_1 - (h_1 + h_2) f_0 + h_1 f_2}{h_1 h_2 (h_1 + h_2)}$$

$$a = \frac{(0.5)(2) - (0.5 + 0.5)(-0.625) + (0.5)(-6)}{(0.5)(0.5)(0.5 + 0.5)} = \frac{1 - (1)(-0.625) + (0.5)(-6)}{(0.5)(0.5)(1)} = -5.5$$

$$b = \frac{f_1 - f_0 - ah_1^2}{h_1}$$

$$b = \frac{2 - (-0.625) - (-5.5)(0.5)^2}{0.5} = \frac{2 + 0.625 + (5.5)(0.25)^2}{0.5} = 8$$

$$x = x_0 - \frac{2c}{b - \sqrt{b^2 - 4ac}}$$

$$x = 0.5 - \frac{2(-0.625)}{8 - \sqrt{(8)^2 - (4)(-5.5)(-0.625)}}$$

$$x = 1.8721$$

#### 2<sup>nd</sup> Iteration

$x_0 = 0.5, x_1 = 1.8721$  and  $x_2 = 1$

$$f(x_0) = f_0 = f(0.5) = (0.5)^2 - 7(0.5)^2 + 14(0.5) - 6 = -0.625$$

$$f(x_1) = f_1 = f(1) = 1.8721^2 - 7(1.8721)^2 + 14(1.8721) - 6 = 2.228$$

$$f(x_2) = f_2 = f(0) = (0)^2 - 7(0)^2 + 14(0) - 6 = -6$$

$$c = f_0 = -0.625$$

$$h_1 = x_1 - x_0 = 1.8721 - 0.5 = 1.3721$$

$$h_2 = x_0 - x_2 = 0.5 - 0 = 0.5$$

$$a = \frac{h_2 f_1 - (h_1 + h_2) f_0 + h_1 f_2}{h_1 h_2 (h_1 + h_2)}$$

$$a = \frac{(-0.5)(2.228) - (1.3721 + 0.5)(-0.625) + (1.3721)(-6)}{(1.3721)(-0.5)(1.3721 + 0.5)} = 14.70$$

$$b = \frac{f_1 - f_0 - ah_1^2}{h_1}$$

$$b = \frac{2.228 - (-0.625) - (14.70)(1.3721)^2}{0.5} = -17.72$$

$$x = x_0 - \frac{2c}{b - \sqrt{b^2 - 4ac}}$$

$$x = 0.5 + \frac{1.25}{12.62 - \sqrt{(17.72)^2 - (4)(14.71)(-0.625)}} = 0.29$$

$$x = 1.8721$$

#### **For 3rd iteration**

$$x_0 = 0.29, x_1 = 1.8721, x_2 = 1$$

For third iteration we will proceed in the same manner.

### Graeffe's Root Square Method

### Linear Equations

### Gaussian Elimination Method

#### **Question 2**

Using Gaussian Elimination Method, solve the following system of equations

$$x_1 + x_2 + 2x_3 = 9$$

$$x_1 + 3x_2 + 2x_3 = 13$$

$$3x_1 + x_2 + 3x_3 = 14$$

**Solution:**

**Marks: 10**

The Augmented Matrix of the given system of equations is

$$A_b = \begin{bmatrix} 1 & 1 & 2 & 9 \\ 1 & 3 & 2 & 13 \\ 3 & 1 & 3 & 14 \end{bmatrix}$$

$$\sim^R \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & 0 & 4 \\ 0 & -2 & -3 & -13 \end{bmatrix} \text{ by } R_2 - R_1 \text{ and } R_3 - 3R_1$$

$$\sim^R \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & -2 & -3 & -13 \end{bmatrix} \text{ by } \frac{1}{2}R_2$$

$$\sim^R \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & -9 \end{bmatrix} \text{ by } R_3 + (-2)R_2$$

$$\sim^R \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \text{ by } \frac{-1}{3}R_3$$

Which shows that from the third and second rows

$$z = 3, y = 2$$

And from the first row

$$x + y + 2z = 9$$

Using the values of  $y$  and  $z$ , we get  $x = 1$

Hence the solution of the given system is

$$\mathbf{x = 1, y = 2, z = 3}$$

#### Question 1

Using Gaussian Elimination Method, solve the following system of equations

$$2x - y + 2z = 2$$

$$x + 10y - 3z = 5$$

$$x - y - z = 3$$

**Marks: 10**

#### Solution:

The Augmented Matrix of the given system of equations is

$$A_b = \begin{bmatrix} 2 & -1 & 2 & 2 \\ 1 & 10 & -3 & 5 \\ 1 & -1 & -1 & 3 \end{bmatrix}$$

$$\sim^R \begin{bmatrix} 1 & 10 & -3 & 5 \\ 2 & -1 & 2 & 2 \\ 1 & -1 & -1 & 3 \end{bmatrix} \text{ by } R_{12}$$

$$\sim^R \begin{bmatrix} 1 & 10 & -3 & 5 \\ 0 & -21 & 8 & -8 \\ 0 & -11 & 2 & -2 \end{bmatrix} \text{ by } R_2 - 2R_1, R_3 - R_1$$

$$\sim^R \begin{bmatrix} 1 & 10 & -3 & 5 \\ 0 & 1 & -\frac{8}{21} & \frac{8}{21} \\ 0 & -11 & 2 & -2 \end{bmatrix} \text{ by } \frac{-1}{21}R_2$$

$$\sim^R \begin{bmatrix} 1 & 10 & -3 & 5 \\ 0 & 1 & -\frac{8}{21} & \frac{8}{21} \\ 0 & 0 & -\frac{46}{21} & \frac{46}{21} \end{bmatrix} \text{ by } R_3 + 11R_2$$

Using Gaussian Elimination method, by backward substitution, we get as follows  
From the third row, we get

$$-\frac{46}{21}z = \frac{46}{21}$$

$$\Rightarrow z = -1$$

From the second row, we get

$$y - \frac{8}{21}z = \frac{8}{21}$$

$$\Rightarrow y = \frac{8}{21} + \frac{8}{21}z$$

Putting the value of  $z$ , we get

$$y = \frac{8}{21} + \frac{8}{21}(-1)$$

$$= \frac{8}{21} - \frac{8}{21}$$

$$= 0$$

And finally from the first row, we get

$$x + 10y - 3z = 5$$

Putting the values of  $y$  and  $z$ , we get

$$x+10(0)-3(-1)=5$$

$$\Rightarrow x+3=5$$

$$\Rightarrow x=2$$

So, the solution is

$$x = 2, y = 0, z = -1$$

### Guass-Jordan Elimination Method

#### Question#1

Marks 10

By using Gauss-Jordan elimination method, solve the following system of equations,

$$x+y+z=7$$

$$3x+3y+4z=24$$

$$2x+y+3z=16$$

#### Solution:

The given system in matrix form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 24 \\ 16 \end{bmatrix}$$
$$A \quad X = B$$

$$\therefore [A|B] = \begin{bmatrix} 1 & 1 & 1 & : & 7 \\ 3 & 3 & 4 & : & 24 \\ 2 & 1 & 3 & : & 16 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 7 \\ 2 & 1 & 3 & : & 16 \\ 3 & 3 & 4 & : & 24 \end{bmatrix} \quad R_{23}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 7 \\ 0 & -1 & 1 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix} \quad R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & : & 9 \\ 0 & -1 & 1 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix} \quad R_1 \rightarrow R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & -1 & 0 & : & -1 \\ 0 & 0 & 1 & : & 3 \end{bmatrix} \quad R_1 \rightarrow R_1 - 2R_3, R_2 \rightarrow R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 3 \end{bmatrix} \quad -R_2$$

So,  
 $x = 3, y = 1, z = 3$

### Jacobi's Iterative Method

Solve the following system of equations

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

By Jacobi's iterative method taking the initial starting of solution vector as  $(0, 0, 0)^T$  and perform the first three iterations.

#### Solution:

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

$$x = \frac{17 - y + 2z}{20}$$

$$y = \frac{-18 - 3x + z}{20}$$

$$z = \frac{25 - 2x + 3y}{20}$$

starting with  $(0, 0, 0)$

#### Iteration#01

$$x = \frac{17 - 0 + 2(0)}{20} = \frac{17}{20} = 0.85$$

$$y = \frac{-18 - 3(0) + 0}{20} = \frac{-18}{20} = -0.9$$

$$z = \frac{25 - 2(0) + 3(0)}{20} = \frac{25}{20} = 1.25$$

#### Iteration#02

$x = 0.85, y = -0.9, z = 1.25$

$$x = \frac{17 - (-0.9) + 2(1.25)}{20} = \frac{20.4}{20} = 1.02$$

$$y = \frac{-18 - 3(0.85) + 1.25}{20} = \frac{-19.3}{20} = -0.965$$

$$z = \frac{25 - 2(0.85) + 3(-0.9)}{20} = \frac{20.6}{20} = 1.03$$

#### Iteration#03

$x = 1.02, y = -0.965, z = 1.03$

$$x = \frac{17 - (-0.965) + 2(1.03)}{20} = \frac{20.025}{20} = 1.00125$$

$$y = \frac{-18 - 3(1.02) + 1.03}{20} = \frac{-20.03}{20} = -1.0015$$

$$z = \frac{25 - 2(1.02) + 3(-0.965)}{20} = \frac{20.065}{20} = 1.00325$$

### Gauss-Seidel Iteration Method

#### Question#3

Marks 10

Solve Question No. #2 by Gauss-Seidel iterative method and perform first three iterations. What you see the difference after solving the same question by two different iterative methods? Give your comments.

#### Solution:

The above system of linear equations is diagonally dominant; therefore, Gauss-Seidel iterative method could be applied to find out real roots

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

The above system of equations could be written in the form

$$x = \frac{17 - y + 2z}{20}$$

$$y = \frac{-18 - 3x + z}{20}$$

$$z = \frac{25 - 2x + 3y}{20}$$

starting with (0,0,0)

#### Iteration#01

$$x = \frac{17 - 0 + 2(0)}{20} = \frac{17}{20} = 0.85$$

$$y = \frac{-18 - 3(0.85) + 0}{20} = \frac{-20.55}{20} = -1.0275$$

$$z = \frac{25 - 2(0.85) + 3(-1.0275)}{20} = \frac{20.2175}{20} = 1.010875$$

#### Iteration#02

$$x = 0.85, y = -1.0275, z = 1.010875$$

$$x = \frac{17 - (-1.0275) + 2(1.010875)}{20} = \frac{20.04925}{20} = 1.0024625$$

$$y = \frac{-18 - 3(1.0024625) + 1.010875}{20} = \frac{-19.9965125}{20} = -0.999825625$$

$$z = \frac{25 - 2(1.0024625) + 3(-0.999825625)}{20} = \frac{19.995598125}{20} = 0.99977990625$$

### Iteration#03

$$x = 1.0024625, y = -0.999825625, z = 0.99977990625$$

$$x = \frac{17 - (-0.999825625) + 2(0.99977990625)}{20} = \frac{19.9904160625}{20} = 0.9999708$$

$$y = \frac{-18 - 3(0.9999708) + 0.99977990625}{20} = \frac{-20.00013249375}{20} = -1.0000066247$$

$$z = \frac{25 - 2(0.9999708) + 3(-1.0000066247)}{20} = \frac{20.000038526}{20} = 1.0000019263$$

In Gauss Seidal Method, the newly computed values in each iteration are directly involved to find the other value of the system of equation and save memory for computation and hence results are more accurate.

### Question 2

Do five iterations to solve the following system of equations by Gauss-Seidal iterative method

$$10x - 2y - 3z = 305$$

$$-2x + 10y - 2z = 154$$

$$-2x - y + 10z = 120$$

**Marks: 10**

**Solution:**

Since, the given system is diagonally dominant; hence we can apply here the Gauss-Seidal method.

From the given system of equations

$$x^{r+1} = \frac{1}{10} [305 + 2y^r + 3z^r]$$

$$y^{r+1} = \frac{1}{10} [154 + 2x^{r+1} + 2z^r]$$

$$z^{r+1} = \frac{1}{10} [120 + 2x^{r+1} + y^{r+1}]$$

### ITERATION 1

For  $r=0$

Taking  $y=z=0$  on right hand side of first equation. In second equation we take  $z=0$  and current value of  $x$ . In third equation we take current value of both  $x$  and  $y$ .

$$x^1 = \frac{1}{10}[305 + 2(0) + 3(0)] = \frac{305}{10} = 30.5$$

$$y^1 = \frac{1}{10}[154 + 2(30.5) + 2(0)] = \frac{1}{10}[154 + 61] = \frac{215}{10} = 21.5$$

$$z^1 = \frac{1}{10}[120 + 2(30.5) + (21.5)] = \frac{1}{10}[120 + 61 + 21.5] = \frac{202.5}{10} = 20.25$$

### ITERATION 2

Similar procedure as used in Iteration 1 will be used for Iterations 2, 3, 4 and 5.

$$x^2 = \frac{1}{10}[305 + 2(21.5) + 3(20.25)] = \frac{408.75}{10} = 40.875$$

$$y^2 = \frac{1}{10}[154 + 2(40.875) + 2(20.25)] = \frac{276.25}{10} = 27.625$$

$$z^2 = \frac{1}{10}[120 + 2(40.875) + (27.625)] = \frac{229.375}{10} = 22.938$$

### ITERATION 3

$$x^3 = \frac{1}{10}[305 + 2(27.625) + 3(22.938)] = \frac{429.064}{10} = 42.906$$

$$y^3 = \frac{1}{10}[154 + 2(42.906) + 2(22.938)] = \frac{285.688}{10} = 28.569$$

$$z^3 = \frac{1}{10}[120 + 2(42.906) + (28.569)] = \frac{234.381}{10} = 23.438$$

### ITERATION 4

$$x^4 = \frac{1}{10}[305 + 2(28.569) + 3(23.438)] = \frac{432.452}{10} = 43.245$$

$$y^4 = \frac{1}{10}[154 + 2(43.245) + 2(23.438)] = \frac{287.366}{10} = 28.737$$

$$z^4 = \frac{1}{10}[120 + 2(43.245) + (28.737)] = \frac{235.227}{10} = 23.523$$

### ITERATION 5

$$x^5 = \frac{1}{10}[305 + 2(28.737) + 3(23.523)] = \frac{433.043}{10} = 43.304$$

$$y^5 = \frac{1}{10}[154 + 2(43.304) + 2(23.523)] = \frac{287.654}{10} = 28.765$$

$$z^5 = \frac{1}{10}[120 + 2(43.304) + (28.765)] = \frac{235.373}{10} = 23.537$$

Above Results are summarized in tabular form as

Iterations	Variables		
	x	y	z
1	30.5	21.5	20.25
2	40.875	27.625	22.938

<b>3</b>	42.906	28.569	23.438
<b>4</b>	43.245	28.737	23.523
<b>5</b>	43.304	28.765	23.537

Hence the solution of the given system of equations, after five iterations, is  
 $x = 43.304$

$$y = 28.765$$

$$z = 23.537$$

Relaxation Method

Matrix Inversion

**Eigen Value Problems**

Power Method

**Question 1**

Find the largest eigen value of the matrix

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

MARKS 10

And the corresponding eigenvector, by Power Method after fourth iteration starting with the initial vector  $v^{(0)} = (0,0,1)^T$

**SOLUTION**

$$\text{Let } A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

Choosing an initial vector as  $v^{(0)} = (0,0,1)^T$  then

**ITERATION 1**

$$u^{(1)} = [A]v^{(0)} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

Now we normalize the resultant vector to get

$$u^{(1)} = 4 \begin{pmatrix} 0 \\ 1 \\ 4 \\ 1 \end{pmatrix} = q_1 v^{(1)}$$

Continuing this procedure for subsequent Iterations , we have

### ITERATION 2

$$u^{(2)} = [A]v^{(1)} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$$

$$u^{(2)} = 6 \begin{pmatrix} \frac{1}{24} \\ 1 \\ \frac{2}{3} \end{pmatrix} = q_2 v^{(2)}$$

### ITERATION 3

$$u^{(3)} = [A]v^{(2)} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{24} \\ 1 \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1.167 \\ 20.708 \\ 2.667 \end{pmatrix}$$

$$u^{(3)} = 20.708 \begin{pmatrix} 0.056 \\ 1 \\ 0.129 \end{pmatrix} = q_3 v^{(3)}$$

### ITERATION 4

$$u^{(4)} = [A]v^{(3)} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0.056 \\ 1 \\ 0.129 \end{pmatrix} = \begin{pmatrix} 1.224 \\ 20.185 \\ 0.516 \end{pmatrix}$$

$$u^{(4)} = 20.185 \begin{pmatrix} 0.061 \\ 1 \\ 0.026 \end{pmatrix} = q_4 v^{(4)}$$

Therefore, the largest eigen value and the corresponding eigen vector accurate to three decimals places are

$$\lambda = 20.185$$

$$(X) = \begin{pmatrix} 0.061 \\ 1 \\ 0.026 \end{pmatrix}$$

=====

**Question 1****marks 10**

Find the largest eigen value of the matrix

$$\begin{bmatrix} 6 & 5 & 2 \\ 3 & 1 & 4 \\ 1 & 10 & 3 \end{bmatrix}$$

And the corresponding eigen vector, by Power Method after fourth iteration starting with the initial vector  $v^{(0)} = (0, 1, 0)^T$ **SOLUTION**

$$\text{Let } A = \begin{pmatrix} 6 & 5 & 2 \\ 3 & 1 & 4 \\ 1 & 10 & 3 \end{pmatrix}$$

Choosing an initial vector

$$v^{(0)} = (0, 1, 0)^T \text{ then}$$

**ITERATION 1**

$$u^{(1)} = [A]v^{(0)} = \begin{pmatrix} 6 & 5 & 2 \\ 3 & 1 & 4 \\ 1 & 10 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 10 \end{pmatrix}$$

Now we normalize the resultant vector to get

$$u^{(1)} = 10 \begin{pmatrix} 0.5 \\ 0.1 \\ 1 \end{pmatrix} = q_1 v^{(1)}$$

Continuing this procedure for subsequent Iterations, we have

**ITERATION 2**

$$u^{(2)} = [A]v^{(1)} = \begin{pmatrix} 6 & 5 & 2 \\ 3 & 1 & 4 \\ 1 & 10 & 3 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5.5 \\ 5.6 \\ 4.5 \end{pmatrix}$$

$$u^{(2)} = 5.6 \begin{pmatrix} \frac{5.5}{5.6} \\ \frac{5.6}{5.6} \\ \frac{4.5}{5.6} \end{pmatrix} = \begin{pmatrix} 0.982 \\ 1 \\ 0.804 \end{pmatrix} = q_2 v^{(2)}$$

### **ITERATION 3**

$$u^{(3)} = [A]v^{(2)} = \begin{pmatrix} 6 & 5 & 2 \\ 3 & 1 & 4 \\ 1 & 10 & 3 \end{pmatrix} \begin{pmatrix} 0.982 \\ 1 \\ 0.804 \end{pmatrix} = \begin{pmatrix} 12.5 \\ 7.162 \\ 13.394 \end{pmatrix}$$

$$u^{(3)} = 13.394 \begin{pmatrix} \frac{0.933}{1} \\ \frac{0.535}{1} \\ 1 \end{pmatrix} = q_3 v^{(3)}$$

### **ITERATION 4**

$$u^{(4)} = [A]v^{(3)} = \begin{pmatrix} 6 & 5 & 2 \\ 3 & 1 & 4 \\ 1 & 10 & 3 \end{pmatrix} \begin{pmatrix} 0.933 \\ 0.535 \\ 1 \end{pmatrix} = \begin{pmatrix} 10.273 \\ 7.334 \\ 9.283 \end{pmatrix}$$

$$u^{(4)} = 10.273 \begin{pmatrix} 1 \\ \frac{0.714}{1} \\ \frac{0.904}{1} \end{pmatrix} = q_4 v^{(4)}$$

Therefore, the largest eigen value and the corresponding eigen vector accurate to four decimals places are

$$\lambda = 10.273$$

and

$$(X) = \begin{pmatrix} 1 \\ 0.714 \\ 0.904 \end{pmatrix}$$

## Jacobi's Method

### Question 1

Using Jacobi's method, find all the eigenvalues and the corresponding eigenvectors of the following matrix,

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Marks: 10

Note: Give results at the end of third rotation.

**Solution.**

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\text{Then, } A' = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

**Hence,**

Matrix A is real and Symmetric and Jacobi's method can be applied.

### Rotation 1

In Matrix A, all the off-diagonal elements are found to be 2.

So, the largest off-diagonal element is found to be

$$a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 2.$$

Therefore, we can choose any one of them as the largest element.

Suppose, we choose  $a_{12}$  as the largest element

Then, we compute the rotation angle as,

$$\tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2 \times 2}{1 - 1} = \infty$$

Therefore

$$\tan 2\theta = \infty$$

$$\Rightarrow 2\theta = \frac{\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Therefore, we construct an orthogonal matrix  $S_1$  such that,

$$S_1 = \begin{bmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} & 0 \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now the first rotation gives,

$$D_1 = S_1^{-1} A S_1$$

Here,  $S_1^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So,

$$D_1 = S_1^{-1} A S_1$$

i.e.

$$D_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 2.828 \\ 0 & -1 & 0 \\ 2.828 & 0 & 1 \end{bmatrix}$$

To check that we are right in our calculations, we can see that the sum of the diagonal elements is,  $3 + (-1) + (1) = 3$ , which is same as the sum of the diagonal elements of the original matrix A.

### **Rotation 2**

For the second rotation we choose the largest off-diagonal element  $d_{13} = d_{31} = 2.828$ , then

$$\tan 2\theta = \frac{2d_{13}}{d_{11} - d_{33}} = \frac{2(2.828)}{3 - (1)} = \frac{5.657}{2} = 2.828$$

So,

$$2\theta = \tan^{-1}(2.828) = 70.526$$

$$\theta = 35.263$$

Therefore, we construct an orthogonal matrix  $S_2$  such that,

$$S_2 = \begin{bmatrix} \cos(35.263) & 0 & -\sin(35.263) \\ 0 & 1 & 0 \\ \sin(35.263) & 0 & \cos(35.263) \end{bmatrix}$$

$$= \begin{bmatrix} 0.817 & 0 & -0.577 \\ 0 & 1 & 0 \\ 0.577 & 0 & 0.817 \end{bmatrix}$$

$$S_2^{-1} = \begin{bmatrix} 0.817 & 0 & 0.577 \\ 0 & 1 & 0 \\ -0.577 & 0 & 0.817 \end{bmatrix}$$

Now the rotation 2 gives,

$$D_2 = S_2^{-1} D_1 S_2$$

$$D_2 = \begin{bmatrix} 0.817 & 0 & 0.577 \\ 0 & 1 & 0 \\ -0.577 & 0 & 0.817 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2.828 \\ 0 & -1 & 0 \\ 2.828 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.817 & 0 & -0.577 \\ 0 & 1 & 0 \\ 0.577 & 0 & 0.817 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Again to check that we are right in our calculations, we can see that the sum of the diagonal elements is  $5 + (-1) + (-1) = 3$  which is same as the sum of the diagonal elements of the original matrix A.

### Rotation 3

We can see that in above iteration that  $D_2$  is a diagonal matrix, so we stop here and take the **Eigen Values** as

$$\lambda_1 = 5, \lambda_2 = -1, \lambda_3 = -1$$

Now the Eigenvectors are the columns vectors of the matrix  $S = S_1 S_2$ , which are,

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.817 & 0 & -0.577 \\ 0 & 1 & 0 \\ 0.577 & 0 & 0.817 \end{bmatrix}$$

$$= \begin{bmatrix} 0.577 & -0.707 & -0.408 \\ 0.577 & 0.707 & -0.408 \\ 0.577 & 0 & 0.817 \end{bmatrix}$$

Therefore, the corresponding **EigenVectors** are

$$X_1 = \begin{bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -0.707 \\ 0.707 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -0.408 \\ -0.408 \\ 0.817 \end{bmatrix}$$

### Interpolation

For a given table of values  $(x_i, y_i)$ ,  $i = 0, 1, 2, \dots, k$  with equally spaced abscissas of a function  $y = f(x)$ , we define the forward difference operator  $\Delta$  as follows,

$$\Delta y_i = y_{i+1} - y_i, \quad i = 0, 1, \dots, (n-1)$$

These differences are called first differences of the function  $y$  and are denoted by the symbol  $\Delta y_i$ . Here,  $\Delta$  is called the first difference operator.

Similarly,  $r$ th Difference operator would be  $\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i$

Leading term  $= y_0$

Leading Difference  $= \Delta y$

Backward Difference Operators:

$$\nabla y_i = y_i - y_{i-1}, \quad i = n, (n-1), \dots, 1$$

Central Difference is given by,

$$\delta y_i = y_{i+(1/2)} - y_{i-(1/2)}$$

$$\begin{aligned}\Delta y_0 &= y_1 - y_0 \\ \Delta y_1 &= y_2 - y_1 \\ &\vdots \quad \vdots \quad \vdots \\ \Delta y_{n-1} &= y_n - y_{n-1}\end{aligned}$$

$$\nabla y_i = y_i - y_{i-1} \quad i = n, (n-1), \dots, 1$$

OR

$$\begin{aligned}\nabla y_1 &= y_1 - y_0 \\ \nabla y_2 &= y_2 - y_1 \\ &\vdots \quad \vdots \quad \vdots \\ \nabla y_n &= y_n - y_{n-1}\end{aligned}$$

$$\delta y_{1/2} = y_1 - y_0, \quad \delta y_{3/2} = y_2 - y_1,$$

In General,

$$\delta y_i = y_{i+(1/2)} - y_{i-(1/2)}$$

Higher order differences are defined as follows:

$$\delta^2 y_i = \delta y_{i-(1/2)} - \delta y_{i-(3/2)}$$

$$\delta^n y_i = \delta^{n-1} y_{i+(1/2)} - \delta^{n-1} y_{i-(1/2)}$$

Shift operator, E

Let  $y = f(x)$  be a function of  $x$ , and let  $x$  takes the consecutive values  $x, x+h, x+2h$ , etc.

We then define an operator having the property

$$E f(x) = f(x+h)$$

Thus, when E operates on  $f(x)$ , the result is the next value of the function. Here, E is called the shift operator. If we apply the operator E twice on  $f(x)$ , we get

$$E^2 f(x) = E[E f(x)]$$

$$= E[f(x+h)] = f(x+2h)$$

Thus, in general, if we apply the operator 'E' n times on  $f(x)$ , we get

$$E^n f(x) = f(x+nh)$$

The inverse operator  $E^{-1}$  is defined as

$$E^{-1} f(x) = f(x-h)$$

Similarly

$$E^{-n} f(x) = f(x-nh)$$

**Average Operator,  $\mu$**

it is defined as

$$\mu f(x) = \frac{1}{2} \left[ f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right) \right]$$

$$= \frac{1}{2} [y_{x+(h/2)} + y_{x-(h/2)}]$$

## Differential Operator, $D$

it is defined as

$$\left. \begin{aligned} Df(x) &= \frac{d}{dx} f(x) = f'(x) \\ D^2 f(x) &= \frac{d^2}{dx^2} f(x) = f''(x) \end{aligned} \right\}$$

$$\Rightarrow \Delta = E - 1$$

$$\Rightarrow \nabla = 1 - E^{-1} = \frac{E - 1}{E}$$

$$\delta = E^{1/2} - E^{-1/2}$$

$$\Rightarrow \mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$$

$$hD = \log E$$

Newton Forward Difference Interpolation.

for any real number  $p$ , we have the operator  $E$  such that

$$E^p f(x) = f(x + ph).$$

$$f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^p f(x_0)$$

$$= \left[ 1 + p\Delta + \frac{p(p-1)}{2!} \Delta^2 + \frac{p(p-1)(p-2)}{3!} \Delta^3 + \dots \right] f(x_0)$$

$$f(x_0 + ph) = f(x_0) + p\Delta f(x_0)$$

$$+ \frac{p(p-1)}{2!} \Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!} \Delta^3 f(x_0)$$

$$+ \dots + \frac{p(p-1)\dots(p-n+1)}{n!} \Delta^n f(x_0) + \text{Error}$$

### Forward Differences

#### Question 2

Construct a forward difference table from the following values of  $x$  and  $y$ .

$x$	1.0	1.2	1.4	1.6	1.8	2.0	2.2
$y=f(x)$	2.0	3.5	-1.7	2.3	4.2	6.5	5.7

Marks:

10

**Solution.**

**Forward-difference table**

$x$	$y$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	2						
		1.5					

1.2	3.5		-6.7				
		-5.2		15.9			
1.4	-1.7		9.2		-27.2		
		4		-11.3		41	
1.6	2.3		-2.1		13.8		-60.8
		1.9		2.5		-19.8	
1.8	4.2		0.4		-6		
		2.3		-3.5			
2.0	6.5		-3.1				
		-0.8					
2.2	5.7						

### Backward Differences

#### Question 2

Construct a backward difference table from the following values of  $x$  and  $y$

$x$	1.0	1.2	1.4	1.6	1.8	2.0	2.2
$y=f(x)$	2.0	3.5	-1.7	2.3	4.2	6.5	5.7

Marks:

10

**Solution.**

**Backward-difference table**

$x$	$y$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	2						
		1.5					
1.2	3.5		-6.7				
		-5.2		15.9			
1.4	-1.7		9.2		-27.2		
		4		-11.3		41	
1.6	2.3		-2.1		13.8		-60.8
		1.9		2.5		-19.8	
1.8	4.2		0.4		-6		
		2.3		-3.5			

2.0	6.5		-3.1				
		-0.8					
2.2	5.7						

**Question 2**  
**10**

**marks**

Form a table of forward and backward differences of the function

$$f(x) = x^3 - 3x^2 - 5x - 7 \text{ For } x = -1, 0, 1, 2, 3, 4, 5$$

**SOLUTION**

For the given function, the values of  $y$  for the given values of  $x$  are calculated as

$$\text{for } x = -1$$

$$f(-1) = (-1)^3 - 3(-1)^2 - 5(-1) - 7 = -6$$

$$\text{for } x = 0$$

$$f(0) = (0)^3 - 3(0)^2 - 5(0) - 7 = -7$$

$$\text{for } x = 1$$

$$f(1) = (1)^3 - 3(1)^2 - 5(1) - 7 = -14$$

$$\text{for } x = 2$$

$$f(2) = (2)^3 - 3(2)^2 - 5(2) - 7 = -21$$

$$\text{for } x = 3$$

$$f(3) = (3)^3 - 3(3)^2 - 5(3) - 7 = -22$$

$$\text{for } x = 4$$

$$f(4) = (4)^3 - 3(4)^2 - 5(4) - 7 = -11$$

$$\text{for } x = 5$$

$$f(5) = (5)^3 - 3(5)^2 - 5(5) - 7 = 18$$

So, the table of values of  $x$  and  $y$  is

$x$	-1	0	1	2	3	4	5
$y = f(x)$	-6	-7	-14	-21	-22	-11	18

**Forward difference table**

Forward difference table for the table of values of  $x$  and  $y = f(x)$  is shown below

$x$	$y$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
-1	-6						
		-1					

0	-7		-6				
		-7		6			
1	-14		0		0		
		-7		6		0	
2	-21		6		0		0
		-1		6		0	
3	-22		12		0		
		11		6			
4	-11		18				
		29					
5	18						

### Backward difference table

Backward difference table for the table of values of  $x$  and  $y = f(x)$  is shown below

$x$	$y$	$\nabla y$	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$
-1	-6						
		-1					
0	-7		-6				
		-7		6			
1	-14		0		0		
		-7		6		0	
2	-21		6		0		0
		-1		6		0	
3	-22		12		0		
		11		6			
4	-11		18				
		29					
5	18						

### Divided Differences

#### Question 1 10

marks

For the following table of values, estimate  $f(2.5)$  using Newton's forward difference interpolation formula.

$x$	1	2	3	4	5	6	7	8
$y = f(x)$	1	8	27	64	125	216	343	512

### SOLUTION

#### Forward difference table

Forward difference table for the given values of  $x$  and  $y$  is shown below

$x$	$y$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1	1						
		7					
2	8		12				
		19		6			
3	27		18		0		
		37		6		0	
4	64		24		0		0
		61		6		0	
5	125		30		0		
		91		6			
6	216		36				
		127					
7	343		42				
		169					
8	512						

Newton's forward difference interpolation formula is given by

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0 + \dots + \frac{p(p-1)(p-2)\dots(p-n+1)}{n!} \Delta^n y_0$$

Here

$$p = \frac{x - x_0}{h} = \frac{2.5 - 1}{1} = \frac{1.5}{1} = 1.5$$

And

$$y_0 = 1, \Delta y_0 = 7, \Delta^2 y_0 = 12, \Delta^3 y_0 = 6, \Delta^4 y_0 = \Delta^5 y_0 = \Delta^6 y_0 = 0$$

So, by putting the above values in Newton's forward difference interpolation formula,  
We have

$$\begin{aligned}
 y_x &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 \\
 &= 1 + 1.5(7) + \frac{1.5(1.5-1)}{2!} (12) + \frac{1.5(1.5-1)(1.5-2)}{3!} (6) \\
 &= 1 + 10.5 + 4.5 - 0.375 \\
 &= 15.625
 \end{aligned}$$

i.e.

$$y_{2.5} = 15.625$$

=====

**Question 2**  
**10**

**marks**

Compute  $f(1.5)$  for the following data by using Newton's divided difference interpolation formula.

$x$	1	2	4	5	8
$f(x)$	5	14	28	45	92

**SOLUTION**

The divided difference table for the given data is given by

$x$	$y$	1 <sup>st</sup> D.D.	2 <sup>nd</sup> D.D.	3 <sup>rd</sup> D.D.	4 <sup>th</sup> D.D.
1	5				
		9			
2	14		-2/3		
		7		1	
4	28		10/3		-29/126
		17		-11/18	
5	45		-1/3		
		47/3			
8	92				

Newton's Divided Difference formula is

$$f(x) = y_0 + (x - x_0)y[x_0, x_1] + (x - x_0)(x - x_1)y[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)y[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3)y[x_0, x_1, x_2, x_3, x_4]$$

$$\begin{aligned} f(1.5) &= 5 + (1.5 - 1)9 + (1.5 - 1)(1.5 - 2)\left(-\frac{2}{3}\right) + (1.5 - 1)(1.5 - 2)(1.5 - 4)1 \\ &\quad + (1.5 - 1)(1.5 - 2)(1.5 - 4)(1.5 - 5)\left(-\frac{29}{126}\right) \\ &= 5 + 4.5 + 0.1667 + 0.625 + 0.5035 \\ &= 10.7952 \end{aligned}$$

**Question 3**

**marks**

**10**

Find  $y'(6)$  and  $y''(6)$  from the following table of values.

$x$	1	2	3	4	5	6
$y$	3	9	17	27	40	55

**SOLUTION**

**Backward difference table**

Backward difference table for the given values of  $x$  and  $y$  is shown below

$x$	$y$	$\nabla y$	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1	3					
		6				
2	9		2			
		8		0		
3	17		2		1	
		10		1		-3
4	27		3		-2	
		13		-1		
5	40		2			
		15				
6	55					

By backward difference formula for first derivative, we have

$$y'_n = \frac{1}{h} \left( \nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} \right)$$

$$y'_n = \frac{1}{h} \left( \nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \frac{\nabla^5 y_n}{5} \right)$$

$$\begin{aligned} y'(6) &= \frac{1}{1} \left( 15 + \frac{2}{2} + \frac{-1}{3} + \frac{-2}{4} + \frac{-3}{5} \right) \\ &= 15 + 1 - 0.333 - 0.5 - 0.6 \\ &= 14.567 \end{aligned}$$

By backward difference formula for second derivative, we have

$$y''_n = \frac{1}{h^2} \left( \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \frac{5}{6} \nabla^5 y_n \right)$$

Therefore

$$\begin{aligned} y''(6) &= \frac{1}{1^2} \left[ 2 + (-1) + \frac{11}{12}(-2) + \frac{5}{6}(-3) \right] \\ &= 2 - 1 - 1.833 - 2.5 \\ &= -3.333 \end{aligned}$$

### Question 1

#### Marks 10

Find the interpolating polynomial for the following data by

- Newton's Forward Difference Formula
- Lagrange's Formula

Hence show that both the methods give rise to the same polynomial.

x	0	1	2	3
y	1	3	7	13

**Solution:**

By Newton's Forward Difference Formula

The Newton's Forward difference table is given as

$x$	$y$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

We have Newton's forward difference interpolation formula as

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots \quad (1)$$

Since, the third and higher order differences are zero; Therefore, Newton's forward difference Interpolation formula reduces to

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 \quad (2)$$

Here,

$$x_0 = 0, y_0 = 1, \Delta y_0 = 2, \Delta^2 y_0 = 2, h = 1$$

And  $p$  is given by

$$p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

Substituting these values in Eq.(2), we have

$$y = 1 + x(2) + \frac{x(x-1)}{2}(2)$$

$$= 1 + 2x + x(x-1)$$

$$= 1 + 2x + x^2 - x$$

$$= x^2 + x + 1$$

Hence, the required polynomial is

$$y = x^2 + x + 1.$$

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b) By Lagrange's Formula

Lagrange's Interpolation formula is given by

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

By putting the values in the formula, we get

$$y = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} (1) + \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} (3)$$

$$+ \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} (7) + \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} (13)$$

$$y = \frac{x^3 - 6x^2 + 11x - 6}{(-1)(-2)(-3)} + \frac{3x^3 - 15x^2 + 18x}{(1)(-1)(-2)} \quad y = \frac{x^3 - 6x^2 + 11x - 6}{-6} + \frac{3x^3 - 15x^2 + 18x}{2}$$

$$+ \frac{7(x^3 - 4x^2 + 3x)}{(2)(1)(-1)} + \frac{13(x^3 - 3x^2 + 2x)}{(3)(2)(1)} \quad + \frac{7x^3 - 28x^2 + 21x}{-2} + \frac{13x^3 - 39x^2 + 26x}{6}$$

$$y = \frac{x^3 - 6x^2 + 11x - 6}{-6} + \frac{3x^3 - 15x^2 + 18x}{2} + \frac{7x^3 - 28x^2 + 21x}{-2} + \frac{13x^3 - 39x^2 + 26x}{6}$$

$$= \frac{-x^3 + 6x^2 - 11x + 6 + 9x^3 - 45x^2 + 54x - 21x^3 + 84x^2 - 63x + 13x^3 - 39x^2 + 26x}{6}$$

$$y = \frac{-x^3 + 9x^3 - 21x^3 + 13x^3 + 6x^2 - 45x^2 + 84x^2 - 39x^2 - 11x + 54x - 63x + 26x + 6}{6}$$

$$= \frac{6x^2 + 6x + 6}{6}$$

$$= \frac{6(x^2 + x + 1)}{6}$$

$$= x^2 + x + 1$$

Thus we have the equation

$$y = x^2 + x + 1$$

This is the same polynomial as obtained in Newton's Forward difference interpolating formula. Hence it is proved that both the methods give rise to the same polynomial

Langrange's Interpolation

Differentiation Using Difference Operators

## Numerical Integration

Trapezoidal Rule

Simpson's 1/3 and 3/8 rules

## Differential Equations

Taylor Series Method

Euler Method

Runge-Kutta Method

Milne's Predictor Corrector Method

Adam Moulton's Predictor Corrector Method

## FAQ updated version.

### Question: What is Bracketing method?

Answer: Methods such as bisection method and the false position method of finding roots of a nonlinear equation  $f(x) = 0$  require bracketing of the root by two guesses. Such methods are called bracketing methods. These methods are always convergent since they are based on reducing the interval between the two guesses to zero in on the root.

### Question: What is an Open method?

Answer: In the Newton-Raphson method, the root is not bracketed. Only one initial guess of the root is needed to get the iterative process started to find the root of an equation. Hence, the method falls in the category of open methods.

### Question: Explain Muller's method briefly.

Answer: In Muller's method,  $f(x) = 0$  is approximated by a second degree polynomial; that is by a quadratic equation that fits through three points in the vicinity of a root. The roots of this quadratic equation are then approximated to the roots of the equation  $f(x) = 0$ . This method is iterative in nature and does not require the evaluation of derivatives as in Newton-Raphson method. This method can also be used to determine both real and complex roots of  $f(x) = 0$ .

### Question: Explain the difference between the linear and non-linear equations.

Answer: Linear Equation An algebraic equation is said to be linear in which each term is either a constant or the product of a constant and the first power of a single variable. One or more variables can be involved in the linear equations. e.g.  $x+3y+z=0$   $2x-y+4z=7$  etc. Non-Linear Equation An equation is said to Non-Linear equation if it is not linear. Equations involving the power of the variable 2 or higher, transcendental,

logarithmic and trigonometric equations etc lie in the category of Non-Linear equations.  
e.g.  $x^2+5x+3=0$   $\sin x+3y+9=0$   $x \log x-7x+4y=2$  etc.

**Question: Explain which value is to be chosen as  $x_0$  in N-R method.**

Answer: If, for a given function,  $f(a) \cdot f(b) < 0$ , then any value between a and b inclusive can be chosen as  $x_0$ .

**Question: Define iterative method of solving linear equations with two examples.**

Answer: Under iterative methods, the initial approximate solution is assumed to be known and is improved towards the exact solution in an iterative way. We consider Jacobi, Gauss-Seidel and relaxation methods under iterative methods.

**Question: Define Pivoting.**

Answer: The Gaussian elimination method fails if any one of the pivot elements becomes zero. In such a situation, we rewrite the equations in a different order to avoid zero pivots. Changing the order of equations is called pivoting.

**Question: Write the two steps of solving the linear equations using Gaussian Elimination method.**

Answer: In this method, the solution to the system of equations is obtained in two stages.

- i) the given system of equations is reduced to an equivalent upper triangular form using elementary transformations
- ii) the upper triangular system is solved using back substitution procedure

**Question: Describe Gauss-Jordan elimination method briefly**

Answer: This method is a variation of Gaussian elimination method. In this method, the elements above and below the diagonal are simultaneously made zero. That is a given system is reduced to an equivalent diagonal form using elementary transformations. Then the solution of the resulting diagonal system is obtained. Sometimes, we normalize the pivot row with respect to the pivot element, before elimination. Partial pivoting is also used whenever the pivot element becomes zero.

**Question: Describe briefly Crout's reduction method.**

Answer: Here the coefficient matrix  $[A]$  of the system of equations is decomposed into the product of two matrices  $[L]$  and  $[U]$ , where  $[L]$  is a lower-triangular matrix and  $[U]$  is an upper-triangular matrix with 1's on its main diagonal.

**Question: Describe briefly the Jacobi's method of solving linear equations.**

Answer: This is an iterative method, where initial approximate solution to a given system of equations is assumed and is improved towards the exact solution in an iterative way.

**Question: What is the difference between Jacobi's method and Gauss Seidel method?**

Answer: The difference between Jacobi's method and Gauss Seidel method is that in Jacobi's method the approximation calculated are used in the next iteration for next approximation but in Gauss-Seidel method the new approximation calculated is instantly replaced by the previous one.

**Question: What is the basic idea of Relaxation method?**

Answer: We can improve the solution vector successively by reducing the largest residual to zero at that iteration. This is the basic idea of relaxation method.

**Question: How the fast convergence in the relaxation method is achieved?**

Answer: To achieve the fast convergence of the procedure, we take all terms to one side and then reorder the equations so that the largest negative coefficients in the equations appear on the diagonal.

**Question: Which matrix will have an inverse?**

Answer: Every square non-singular matrix will have an inverse.

**Question: What are the popular methods available for finding the inverse of a matrix?**

Answer: Gauss elimination and Gauss-Jordan methods are popular among many methods available for finding the inverse of a matrix.

**Question: Explain Gaussian Elimination Method for finding the inverse of a matrix.**

Answer: In this method, if  $A$  is a given matrix, for which we have to find the inverse; at first, we place an identity matrix, whose order is same as that of  $A$ , adjacent to  $A$  which we call an augmented matrix. Then the inverse of  $A$  is computed in two stages. In the first stage,  $A$  is converted into an upper triangular form, using Gaussian elimination method. In the second stage, the above upper triangular matrix is reduced to an identity matrix by row transformations. All these operations are also performed on the adjacently placed identity matrix. Finally, when  $A$  is transformed into an identity matrix, the adjacent matrix gives the inverse of  $A$ . In order to increase the accuracy of the result, it is essential to employ partial pivoting.

**Question: What are the steps for finding the largest eigen value by power method.**

Answer:

Procedure Step 1: Choose the initial vector such that the largest element is unity.

Step 2: The normalized vector is pre-multiplied by the matrix  $[A]$ .

Step 3: The resultant vector is again normalized.

**Question: What is the method for finding the eigen value of the least magnitude of the matrix  $[A]$ ?**

Answer: For finding the eigen value of the least magnitude of the matrix  $[A]$ , we have to apply power method to the inverse of  $[A]$ .

**Question: What is interpolation?**

Answer: The process of estimating the value of  $y$ , for any intermediate value of  $x$ , lying inside the table of values of  $x$  is called interpolation.

**Question: What is extrapolation?**

Answer: The method of computing the value of  $y$ , for a given value of  $x$ , lying outside the table of values of  $x$  is known as extrapolation.

**Question: What happens when shift operator  $E$  operates on the function.**

Answer: When shift operator  $E$  operates on the function it results in the next value of the function.  $Ef(x) = f(x+h)$

**Question: What is the basic condition for the data to apply Newton's interpolation methods?**

Answer: To apply Newton's interpolation methods, data should be equally spaced.

**Question: When is the Newton's forward difference interpolation formula used?**

Answer: Newton's forward difference interpolation formula is mainly used for interpolating the values of  $y$  near the beginning of a set of tabular values and for extrapolating values of  $y$ , a short distance backward from  $y_0$ .

**Question: When is the Newton's backward difference interpolation formula used?**

Answer: Newton's backward difference interpolation formula is mainly used for interpolating the values of  $y$  near the end of a set of tabular values and for extrapolating values of  $y$ , a short distance forward from  $y_n$ .

**Question: When the Newton's backward difference interpolation formula is used?**

Answer: For interpolating the value of the function  $y = f(x)$  near the end of table of values, and to extrapolate value of the function a short distance forward from  $y_n$ , Newton's backward Interpolation formula is used.

**Question: What is the formula for finding the value of  $p$  in Newton's forward difference interpolation formula?**

Answer:  $P = (x - x_0) / h$

**Question: What is the formula for finding the value of  $p$  in Newton's backward difference interpolation formula?**

Answer:  $P = (x - x_n) / h$

**Question: If the values of the independent variable are not equally spaced then which formula should be used for interpolation?**

Answer: If the values of the independent variable are not given at equidistant intervals, then the Lagrange's interpolation formula should be used.

**Question: To use Newton's divided difference interpolation formula, what should the values of independent variables be?**

Answer: To use Newton's divided difference interpolation formula, the values of independent variables should not be equally spaced.

**Question: Which difference formula is symmetric function of its arguments?**

Answer: Divided difference formula is symmetric function of its arguments.

**Question: Is the interpolating polynomial found by Lagrange's and Newton's divided difference formulae is same?**

Answer: Yes. The interpolating polynomial found by Lagrange's and Newton's divided difference formulae is one and the same.

**Question: Which formula involves less number of arithmetic operations? Newton or Lagrange's?**

Answer: Newton's formula involves less number of arithmetic operations than that of Lagrange's.

**Question: When do we need Numerical Methods for differentiation and integration?**

Answer: If the function is known and simple, we can easily obtain its derivative (s) or can evaluate its definite integral. However, if we do not know the function as such or the function is complicated and is given in a tabular form at a set of points  $x_0, x_1, \dots, x_n$ , we use only numerical methods for differentiation or integration of the given function.

**Question:** If the value of the independent variable at which the derivative is to be found appears at the beginning of the table of values, then which formula should be used?

**Answer:** If the value of the independent variable at which the derivative is to be found appears at and near the beginning of the table, it is appropriate to use formulae based on forward differences to find the derivatives.

**Question:** If the value of the independent variable at which the derivative is to be found occurs at the end of the table of values, then which formula should be used?

**Answer:** If the value of the independent variable at which the derivative is to be found occurs at the end of the table of values, it is appropriate to use formulae based on backward differences to find the derivatives.

**Question:** Why we need to use RICHARDSON'S EXTRAPOLATION METHOD?

**Answer:** To improve the accuracy of the derivative of a function, which is computed by starting with an arbitrarily selected value of  $h$ , Richardson's extrapolation method is often employed in practice.

**Question:** To apply Simpson's 1/3 rule, what should the number of intervals be?

**Answer:** To apply Simpson's 1/3 rule, the number of intervals must be even.

**Question:** To apply Simpson's 3/8 rule, what should the number of intervals be?

**Answer:** To apply Simpson's 3/8 rule, the number of intervals must be multiple of 3.

**Question:** What is the order of global error in Simpson's 1/3 rule?

**Answer:** The global error in Simpson's 1/3 rule is of the order of  $O(h^4)$ .

**Question:** Is the order of global error in Simpson's 1/3 rule equal to the order of global error in Simpson's 3/8 rule?

**Answer:** Yes. The order of global error in Simpson's 1/3 rule equal to the order of global error in Simpson's 3/8 rule.

**Question:** What is the order of global error in Trapezoidal rule?

**Answer:** The global error in Trapezoidal rule is of the order of  $O(h^2)$ .

**Question:** What is the formula for finding the width of the interval?

**Answer:** Width of the interval,  $h$ , is found by the formula  $h=(b-a)/n$

**Question:** What type of region does the double integration give?

**Answer:** Double integration gives the area of the rectangular region.

**Question:** Compare the accuracy of Romberg's integration method to trapezoidal and Simpson's rule.

**Answer:** Romberg's integration method is more accurate to trapezoidal and Simpson's rule

**Question:** What is the order of global error in Simpson's 3/8 rule?

**Answer:** The global error in Simpson's 3/8 rule is of the order of  $O(h^4)$ .

**Question:** Which equation models the rate of change of any quantity with respect to another?

**Answer:** An ordinary differential equation models the rate of change of any quantity with respect to another.

**Question:** By employing which formula, Adam-Moulton P-C method is derived?

**Answer:** Adam-Moulton P-C method is derived by employing Newton's backward difference interpolation formula.

**Question:** What are the commonly used number systems in computers?

Answer: Binary Octal Decimal Hexadecimal

**Question:** If a system has the base  $M$ , then how many different symbols are needed to represent an arbitrary number? Also name those symbols.

Answer:  $M - 1$ ,  $0, 1, 2, 3, \dots, M-1$

**Question:** What is inherent error and give its example.

Answer: It is that quantity of error which is present in the statement of the problem itself, before finding its solution. It arises due to the simplified assumptions made in the mathematical modeling of a problem. It can also arise when the data is obtained from certain physical measurements of the parameters of the problem. e.g. if writing 8375 instead of 8379 in a statement lies in the category of inherent error.

**Question:** What is local round-off error?

Answer: At the end of computation of a particular problem, the final results in the computer, which is obviously in binary form, should be converted into decimal form—a form understandable to the user—before their print out. Therefore, an additional error is committed at this stage too. This error is called local round-off error.

**Question:** What is meant by local truncation error?

Answer: Retaining the first few terms of the series up to some fixed terms and truncating the remaining terms arise to local truncation error.

**Question:** What is transcendental equation and give two examples.

Answer: An equation is said to be transcendental equation if it has logarithmic, trigonometric and exponential function or combination of all these three. For example it is a transcendental equation as it has an exponential function. These all are the examples of transcendental equation.

**Question:** What is meant by intermediate value property?

Answer: If  $f(x)$  is a real valued continuous function in the closed interval  $[a, b]$  and  $f(a)$  and  $f(b)$  have opposite signs, then  $f(x)=0$  has at least one root such that  $a < x < b$ . Simply if  $f(x)=0$  is a polynomial equation and if  $f(a)$  and  $f(b)$  are of different signs, then  $f(x)=0$  must have at least one real root between  $a$  and  $b$ .

**Question:** What is direct methods of solving equations?

Answer: Those methods which do not require any information about the initial approximation of root to start the solution are known as direct methods. The examples of direct methods are Graeffe root squaring method, Gauss elimination method and Gauss Jordan method. All these methods do not require any type of initial approximation.

**Question:** What is Iterative method of solving equations?

Answer: These methods require an initial approximation to start. Bisection method, Newton Raphson method, secant method, Jacobi method are all examples of iterative methods.

**Question:** If an equation is transcendental, then in which mode the calculations should be done?

Answer: All the calculations in the transcendental equations should be done in the radian mode.

**Question: What is the convergence criterion in method of iteration?**

Answer: If  $x$  be a root of  $f(x) = 0$  which is equivalent to  $I$  be any interval containing the point  $x =$  and will converge to the root provided that the initial approximation is chosen in  $I$

**Question: When we stop doing iterations when TOL is given?**

Answer: Here if TOL is given then we can simply find the value of TOL by subtracting both the consecutive roots and write it in the exponential notation if the required TOL is obtained then we stop.

**Question: How the value of h is calculated in interpolation?**

Answer: There are two types of data in the interpolation one is equally spaced and other is unequally spaced. In equally spaced data we need to calculate the value of  $h$  that is calculated by subtracting any two consecutive values and taking their absolute value.

**Question: What is an algebraic equation?**

Answer: An algebraic equation is an equation which is purely polynomial in any variable. Supposed  $x^2+3x+2=0$ ,  $x^4+3x^2=0$ ,  $y^3+6y^2=0$  all are algebraic equations as these are purely polynomial in  $x$  and  $y$  variable.

**Question: What is Descartes rule of signs?**

Answer: The number of positive roots of an algebraic equation  $f(x) = 0$  can not exceed the no of changes in signs. Similarly the no of negative roots of and algebraic equation can not exceed the no of changes in sign of equation  $f(-x) = 0$ .

**Question: What are direct methods?**

Answer: The numerical methods which need no information about the initial approximation are known as direct methods like Graffee's root squaring method.

**Question: What is meant by iterative methods?**

Answer: The methods which need one or more iterations are known as iterative methods like bisection method, Newton raphson method, and many other methods.

**Question: What is graphically meant by the root of the equation?**

Answer: If the graph of a function  $f(x) = 0$  cuts the  $x$ -axis at a point  $a$  then  $a$  is known as the root of the equation.

**Question: Q. What is the difference between open and bracketing method?**

Answer: In open methods we need only one initial approximation of the root that may be any where lying and if it is not very close then we have to perform more iteration and the example of open method is Newton raphson method. In bracketing method we bracket the root and find that interval in which root lies means we need two initial approximations for the root finding. Bisection method is an example of the bracketing method.

**Question: Condition for the existence of solution of the system of equations.**

Answer: If the  $|A|$  is not equal to zero then the system will have a unique solution if  $|A| = 0$  then the system will have no solution

**Question: Should the system be diagonally dominant for gauss elimination method?**

Answer: The system of equation need not to be diagonally dominant for Gauss elimination method and gauss Jordan method for both the direct method it is not necessary for the system to be diagonally dominant .it should be diagonally dominant for iterative methods like jacobie and gauss seidel method.

**Question: What is meant by diagonally dominant system?**

Answer: A system  $a_1x+b_1y+c_1z=d_1$ ,  $a_2x+b_2y+c_2z=d_2$ ,  $a_3x+b_3y+c_3z=d_3$  is said to be diagonally dominant if the following condition holds.  $|a_1| \Rightarrow$  (greater or equal)  $|b_1|+|c_1|$   $|b_2| \Rightarrow$  (greater or equal)  $|a_2|+|c_2|$   $|c_3| \Rightarrow$  (greater or equal)  $|a_3|+|b_3|$

**Question: State the sufficient condition for the convergence of the system of equation by iterative methods.**

Answer: A sufficient condition for convergence of iterative solution to exact solution is  $|a_1| \Rightarrow$  (greater or equal)  $|b_1|+|c_1|$   $|b_2| \Rightarrow$  (greater or equal)  $|a_2|+|c_2|$   $|c_3| \Rightarrow$  (greater or equal)  $|a_3|+|b_3|$  For the system  $a_1x+b_1y+c_1z=d_1$ ,  $a_2x+b_2y+c_2z=d_2$ ,  $a_3x+b_3y+c_3z=d_3$  Similarly for the system with more variables we can also construct the same condition

**Question: The calculation for numerical analysis should be done in degree or radians.**

Answer: All the calculation for numerical analysis should be done in radians not in degrees set your calculator in radians mode and suppose the value of  $\pi=3.14$ .

**Question: How we can identify that Newton forward or backwards interpolation formula is to be used.**

Answer: If the value at which we have to interpolate is in the start of the table then we will use Newton's forward interpolation formula if it is at the end of the table then we will use the Newton's backward interpolation formula.

**Question: What is meant precision and accuracy?**

Answer: Precision and accuracy are two terms which are used in numerical calculations by precision we mean that how the values in different iterations agree to each other or how close are the different values in successive iterations. For example you have performed 3 different iterations and result of all the iteration are 1.32514, 1.32516, 31.32518 these three values are very precise as these values agree with each other. Accuracy means the closeness to the actual value. Suppose that you have calculated an answer after some iteration and the answer is 2.718245 and the actual answer is 2.718254 and the answer calculated is very accurate but if this answer is 2.72125 then it is not accurate.

**Question: What is the condition that a root will lie in an interval.**

Answer: Suppose that you have a function  $f(x)$  and an interval  $[a,b]$  and you calculate both  $f(a)$  and  $f(b)$  if  $f(a)f(b)<0$  then there must exist a root for this function in this interval. In simple words  $f(a)$  and  $f(b)$  must have opposite signs.

**Question: How the divided difference table is constructed?**

Answer:

x	y	First difference	Second difference	Third difference	1	0.2	y <sub>0</sub>
	$y_1 - y_0 / x_1 - x_0 = s_{y0}$	$s_{y1} - s_{y0} / x_2 - x_0 = s_{2y0}$	$s_{2y0} - s_{2y1} / x_3 - x_0 = s_{3y0}$	$s_{3y1} - s_{3y0} / x_4 - x_0 = s_{4y0}$	2	0.69	$y_1$
	$s_{2y1} - s_{2y0} / x_2 - x_1 = s_{y1}$	$s_{y2} - s_{y1} / x_3 - x_1 = s_{2y1}$	$s_{2y2} - s_{2y1} / x_4 - x_1 = s_{3y1}$	3	0.985	$y_2$	
	$s_{3y2} - s_{3y1} / x_3 - x_2 = s_{y2}$	$s_{y3} - s_{y2} / x_4 - x_2 = s_{2y2}$	4	1.2365	$y_3$		
	$s_{4y3} - s_{4y2} / x_4 - x_3 = s_{y3}$	5	2.3651	$y_4$			

This is the complete table showing the central differences. Here s stands for difference

**Question: What is Gauss-seidel Method**

Answer: It is also an iterative method and in this we also check either the system is diagonally dominant and, if so then we proceed on the same lines as in the Jacobi's method. The difference in both is when we calculate a value of any variable that is instantly replaced by the previous value. If we have initial values like  $x_1=0, x_2=0$  and

$x_3=0$  in the case of three variables in the first iteration we will use all the three values to calculate  $x_1$  but to calculate the value of  $x_2$  we will use the recent value of  $x_1$  which is calculated in the first iteration.

**Question: What is partial and full pivoting?**

Answer: Partial and full pivoting, In gauss elimination method when you have any of diagonal element  $a_{ii}$  zero it means the solution does not exist to avoid this we change the equation so that a non zero pivot is achieved. Now you have an argument matrix in which the elements in the first column are 1,3,4 respectively here in case of partial pivoting we will replace first element with the last element it is done by replacing the first equation with last equation it is known as partial pivoting. In full pivoting we change rows and columns but that is not implemented manually it is used in computers.

**Question: How the initial vector is chosen in power method?**

Answer: Choose the initial vector such that the largest element is unity. But no problem with this you will proceed with this vector as it is and will not change it. In the given question the initial vector is provided so you have to proceed with the given vector and will normalize this vector and normalizing this vector you will keep in mind that the greatest value should be unity. An eigenvector  $V$  is said to be normalized if the coordinate of largest magnitude is equal to unity. Actually this condition is for the normalization of the vectors when you have to normalize the vector you keep in mind that the largest entry in the vector must be 1 and so you take the largest element and divide the remaining values by the greatest value.

**Question: what is the relationship between  $p=0$  and non zero  $p$  in interpolation.**

Answer:  $f_p = f_0 + pDf_0 + \frac{1}{2}(p^2 - p)D^2f_0 + \frac{1}{6}(p^3 - 3p^2 + 2p)D^3f_0 + \frac{1}{24}(p^4 - 6p^3 + 11p^2 - 6p)D^4f_0 + \dots$  This is the interpolation formula. For the derivative you will have to take the derivative of this formula w.r.t  $p$  and you will get  $f'_p = \frac{1}{h}(Df_0 + \frac{1}{2}(2p-1)D^2f_0 + \frac{1}{6}(3p^2 - 6p + 2)D^3f_0 + \dots)$  Now put here  $p=0$   $f'_p = \frac{1}{h}(Df_0 + \frac{1}{2}(0-1)D^2f_0 + \frac{1}{6}(0-0+2)D^3f_0 + \dots)$  So the formula becomes  $f'_p = \frac{1}{h}(Df_0 + D^2f_0/2 + D^3f_0/3 + \dots)$  so this is the relationship between non zero and zero  $p$  no when you have to calculate  $p$  the you use formula  $p = (x - x_0)/h$  so this is the impact of  $x$ .

**Question: What is chopping and Rounding off?**

Answer: Chopping and rounding are two different techniques used to truncate the terms needed according to your accuracy needs. In chopping you simply use the mentioned number of digits after the decimal and discard all the remaining terms. Explanation  $(1/3 - 3/11) + 3/20 = (0.333333... - 0.27272727...) + 0.15 = (0.333 - 0.272) + 0.15$  This is the three digit chopping.

**Question: When the forward and backward interpolation formulae are used?**

Answer: In interpolation if we have at the start then we use the forward difference formula and the formula to calculate  $p$  is  $(x - x_0)/h$ . If the value of  $x$  lies at the end then we use Newton's backward formula and formula to calculate the value of  $p$  is  $(x - x_n)/h$ . Now I come to your question as in this case the value lies at the end so  $6$  will be used as the  $x_n$ . This procedure has been followed by the teacher in the lectures. But some authors also use another technique that is if you calculate the value of  $p$  and that is negative then the origin is shifted to that value for which the value of  $p$  becomes positive. And then according to that origin the values of differences are used and you need not follow this procedure.

**Question: What is forward and backward difference operator and the construction of their table.**

Answer: For forward  $D_{fr} = f_{r+1} - f_r$   $Df_0 = f_1 - f_0$  In terms of  $y$   $D_{yr+1} = y_{r+1} - y_r$   $D$  stands for the forward difference operator For backward  $D_{br} = f_r - f_{r-1}$   $Df_1 = f_1 - f_0$  In terms of  $y$   $Dy_1 = y_1 - y_0$  Here  $D$  stands for backwards operator Now the construction of the difference table is based on X Y 1st forward 2nd forward 3rd forward  $x_1$   $Y_1$   $Y_2 - Y_1 = Dy_0$   $x_2$   $Y_2$   $Y_3 - Y_2 = Dy_1$   $x_3$   $Y_3$   $Y_4 - Y_3 = Dy_2$   $x_4$   $Y_4$  Now consider the construction of table for the backward table X Y 1st forward 2nd forward 3rd forward  $x_1$   $Y_1$   $Y_2 - Y_1 = Dy_1$   $x_2$   $Y_2$   $Y_3 - Y_2 = Dy_2$   $x_3$   $Y_3$   $Y_4 - Y_3 = Dy_3$   $x_4$   $Y_4$

Dear student this is the main difference in the construction of the forward and backwards difference table when you proceed for forward difference table you get in first difference the value  $Dy_0$  but in the construction of backwards difference table in the first difference you get  $Dy_1$  and in the second difference in the forward difference table you get  $D^2 y_0$  and in the backward difference table the first value in the second difference is  $D^2 y_1$ . I think so you have made it clear.

**Question: What is Jacobi's method?**

Answer: Jacobi's Method It is an iterative method and in this method we first of all check either the system is diagonally dominant and, if the system is diagonally dominant then we will calculate the value of first variable from first equation in the form of other variables and from the second equation the value of second variable in the form of other variables and so on. We are provided with the initial approximations and these approximations are used in the first iteration to calculate first approximation of all the variables. The approximations calculated in the first iteration are used in the second iteration to calculate the second approximations and so on.

**Question: what is Simpson's 3/8th rule.**

Answer: The general formula for Simpson's 3/8th rule is  $3h/8[f_0 + 3(f_1 + f_2) + 2f_3 + (3f_4 + f_5) + 2f_6 + \dots + 3(f_{n-2} + f_{n-1}) + f_n]$  Now if we have to calculate the integral by using this rule then we can simply proceed just write first and last value and distribute all the remaining values with prefix 3 and 2 Like you have  $f_0, f_1, f_2, f_3, f_4$  Then the integral can be calculated as  $3h/8[f_0 + f_4 + 3(f_1 + f_2) + 2f_3]$  If we have values like  $f_0, f_1, f_2, f_3, f_4, f_5$  Then integral can be calculated as  $3h/8[f_0 + 3(f_1 + f_2) + 2f_3 + 3f_4 + f_5]$  Similarly proceeding in this fashion we can calculate the integral in this fashion

**Question: what is classic runge-kutta method**

Answer: The fourth order Runge-Kutta method is known as the classic formula of classic Runge-Kutta method.  $y_{n+1} = y_n + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$  Where  $k_1 = hf(x_n, y_n)$   $k_2 = hf(x_n + h/2, y_n + k_1/2)$   $k_3 = hf(x_n + h/2, y_n + k_2/2)$   $k_4 = hf(x_n + h, y_n + k_3)$

**Question: What is meant by TOL?**

Answer: The TOL means the extent of accuracy which is needed for the solution. If you need the accuracy to two places of decimal then the TOL will be  $10^{-2}$ . Similarly the  $10^{-3}$  means that the accuracy needed to three places of decimal. Suppose you have the root from last iteration 0.8324516 and 0.8324525 if we subtract both and consider absolute value of the difference 0.0000009 now it can be written as  $0.09 \times 10^{-5}$  so the TOL in this case is  $10^{-5}$ . Similarly if we have been provided that you have to for the TOL  $10^{-2}$  you will check in the same way. In the given equation you will solve the equation by any method and will consider some specific TOL and try to go to that TOL. Some time no TOL is provided and you are asked to perform to some specific no of iterations.

**Question: what is meant by uniqueness of LU method.**

Answer: An invertible (whose inverse exists) matrix can have LU factorization if and only if all its principal minors (the determinant of a smaller matrix in a matrix) are non zero. The factorization is unique if we require that the diagonal of L or U must have 1's. The matrix has a unique LDU factorization under these conditions. If the matrix is singular (inverse does not exist) then an LU factorization may still exist, a square matrix of rank (the rank of a matrix in a field is the maximal no of rows or column)  $k$  has an LU factorization if the first  $k$  principal minors are non zero. These are the conditions for the uniqueness of the LU decomposition.

**Question: how the value of h is calculated from equally spaced data.**

Answer: Consider the following data  $x$   $y$  1 1.6543 2 1.6984 3 2.4546 4 2.9732 5 3.2564 6 3.8765 Here for  $h=2-1=3-2=1$   $x$   $y$  0.1 1.6543 0.2 1.6984 0.3 2.4546 0.4 2.9732 0.5 3.2564 0.6 3.8765 Here for the calculation of  $h=0.2-0.1=0.3-0.2=0.1$  I think so that you may be able to understand.

**Glossary (Updated Version)****Absolute Error :**

The absolute error is used to denote the actual value of a quantity less its rounded value if  $x$  and  $x^*$  are respectively the rounded and actual values of a quantity, then absolute error is defined by  $AE = |x - x^*|$

**Accuracy :**

The extent of the closeness between the actual value and estimated value is known as accuracy. Suppose you have taken readings like 2.1234, 2.1236 and 2.1238 and 2.45, 2.52, 2.63 Now if the actual root is 2.65 then the second values are more accurate and not precise but first set of values is precise but not accurate as it differs from the actual value.

**Algebraic equation :**

The equation  $f(x)=0$  is known as algebraic equation if it is purely a polynomial in  $x$ . Like  $f(x)=3$ ,  $f(x)=x^3-3x-4$  are some examples of algebraic equations and if the variable is changed to  $y$  or any one then under the same conditions it will be algebraic.

**Bisection method :**

It is a bracketing method which is used to locate the root of an equation and it makes use of the intermediate value property to locate the interval in which the root lies, the formula which is used is  $(a+b)/2$  where  $a$  and  $b$  are points such that  $f(a)f(b) < 0$ .

**Bracketing Method :**

The iterative methods which require two initial approximations for its first iteration are known as bracketing methods. Bisection method is the example of the bracketing method as it requires an interval for the approximation of the root.

**Crout's Method :**

This method is used to solve the system of equations by decomposing the system into two matrices  $L$  and  $U$  where  $L$  is a lower triangular matrix and  $U$  is an upper triangular matrix. In  $U$  all the elements in the main diagonal are 1.

Descartes rule of signs : The no of positive roots of an algebraic equation  $f(x)=0$  with real coefficients cannot exceed the no of changes in the in sign of the coefficients in the polynomial  $f(x)=0$ , similarly the no of negative roots of  $f(x)=0$  can not exceed the no of changes in the sign of the coefficients of  $f(-x)=0$

#### Direct methods :

These are the methods which do not need the knowledge of the initial approximation are known as direct methods.

#### Gauss Elimination Method :

It is a direct method which is used to solve a system of equations

#### Gauss Seidel iterative method :

It is an iterative method in which an initial approximation is given. First of all system should be checked either it is diagonally dominant or not if it is not then it is made diagonally dominant. Secondly the first variable is calculated in terms of other variables from first equation and Second variable from second equation and so on .the previous value of the variable is replaced by new value instantly as it is obtained. This is the difference in the gauss seidel and jacobie iterative method.

#### Graeffee's root squaring method :

This method is used to find all the roots of the polynomial equation.

#### Intermediate value property :

If for an equation  $f(x)=0$  for two values  $a$  and  $b$  we have such that  $f(a)f(b)<0$  then there must exist a root between  $a$  and  $b$  in the interval  $[a,b]$ .

#### Inverse of a matrix :

The matrix  $B$  is said to be inverse of a matrix  $A$  if the product of  $A$  and  $B$  is  $I$  identity matrix.

#### Iterative methods :

Iterative methods are those type of methods which always require an initial approximation to start an iteration

#### Jacobie's iterative method: :

It is an iterative method in which an initial approximation is given. First of all system should be checked either it is diagonally dominant or not if it is not then it is made diagonally dominant. Secondly the first variable is calculated in terms of other variables from first equation and Second variable from second equation and so on after first iteration in the second iteration all the variables are replaced by the previous one.

#### Muller's Method :

In Muller's method  $f(x)=0$  is approximated by a second degree polynomial, that is by quadratic equation that fits through three points in the vicinity of a root .Then roots of this quadratic equation are then approximated to the roots of equation  $f(x)=0$

#### Newton Raphson Method. :

It is an open method which is used to locate a root of the equation, it needs only one initial approximation for it's first iteration.  $x_{n+1}=x_n-(f(x_n)/f'(x_n))$

#### Non singular matrix :

A matrix is said to be non singular if the determinant of the matrix is non zero.  $|A|$  is not equal to zero and the inverse of non singular exists.

#### Open methods :

The methods which require only one initial approximation to start the first iteration, for example the Newton's Raphsom method is known as the open method as it requires only one initial approximation.

#### Pivoting :

If any of the pivot elements in gauss elimination become zero then this method fails so to Avoid such type of situation equation are rearranged to get rid of zero pivot element, this procedure is known as pivoting.

Polynomial:

An expression of the form  $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$  where n is a positive integer and  $a_0, a_1, a_2, \dots, a_n$  are real constants, such type of expression is called an nth degree polynomial in x if  $a_0 \neq 0$

#### Precision :

The extent of closeness of different measurement taken to estimate a certain value. Suppose you have done different iterations to measure the root of an equation and take different values as 2.1234, 2.1236, and 2.1238 these all the values are very close to each other so these are very precise.

#### Regula -Falsi method :

It is also an iterative method and is a bracketing method and use intermediate value property to get it's initial guess. The formula used for this is  $x_{n+1} = x_n - (x_n - x_{n-1}) / (f(x_n) - f(x_{n-1}))$

Relative Error : It is the ratio of the absolute error to the actual value of the quantity. Thus  $RE = AE / |x^*|$

#### Relaxation method :

It is also an iterative method and in this method you solve the system of equation by making the greatest residual to zero.

Root : If you have an equation  $f(x)=0$  then the no a is said to be the root of the equation if  $f(a)=0$  Suppose you have an equation  $f(x)=x^2-4$  the 2 is a root of the equation as  $f(2)=4-4=0$

#### Secant Method :

The secant method is also an open method and it takes two initial values for it's first approximation, the formula used for this is known as  $x_{n+1} = (x_n - 1f(x_n) - x_n f(x_{n-1})) / (f(x_n) - f(x_{n-1}))$

#### Significant digits :

A significant digit in an approximate no is a digit, which gives reliable information about the size of the number. In other words a significant digit is used to express accuracy, that is how many digits in the no have meaning.

#### Singular matrix :

A matrix s said to be a singular matrix if the determinant of the matrix is zero  $|A|=0$ , the inverse of the singular matrix do not exist.

### Transcendental equation :

An equation  $f(x) = 0$  is said to be transcendental equation if it contains trigonometric, logarithmic and exponential functions

### Truncation Error :

It is defined as the replacement of one series by another series with fewer terms. The error arising by this approximation is known as truncation error.

### Important Formula For MTH603

#### Bisection Method

$$x_2 = \frac{x_0 + x_1}{2}$$

#### Muller Method

$$x = \frac{x_0 - 2c}{+b - \sqrt{b^2 - 4ac}}$$

in this formula  $x_0, x_1, x_2$  will be and U will just put the values according to the given formulas.

$$a = \frac{h_2 f_1 - (h_1 + h_2) f_0 + h_1 f_2}{h_1 h_2 (h_1 + h_2)}$$

$$b = \frac{f_1 - f_0 - a h_1^2}{h_1}$$

$$c = f(x_0)$$

$$h_1 = x_1 - x_0$$

$$h_2 = x_0 - x_2$$

#### Regula Falsi Method (Method of False position)

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

#### Newton Rophson method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

#### Secant Method

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

### Newton's Formula

$$x_1 = \frac{1}{2}(x_0 + \frac{n}{x_0})$$

in this formula

$x_0 = 2$  perfect square near to 12 such like 9 and 16

### Graeffe root squaring method

$$f(x)f(-x) = a_3^2 x^6 - (a_2^2 - 2a_1 a_3) x^4 + (a_1^2 - 2a_0 a_2) x^2 - x_0^2$$

the truncation error (TE) is given by

$$TE \leq \frac{x^{2n+2}}{(2n+2)!}$$

The TE is independent of the computer used.

### Spring 2011 Latest Papers (Current)

#### Papers Number 01

Total 26 questions

20 mcqs

**Question:** What are the steps for finding the largest eigen value by power method?

**Answer:** Procedure

Step 1: Choose the initial vector such that the largest element is unity.

Step 2: The normalized vector is pre-multiplied by the matrix [A].

Step 3: The resultant vector is again normalized.

Step 4: This process of iteration is continued and the new normalized vector is repeatedly pre-multiplied by the matrix [A] until the required accuracy is obtained.

At this point, the result looks like

$$u^{(k)} = [A]v^{(k-1)} = q_k v^{(k)}$$

Here,  $q_k$  is the desired largest eigen value and  $v^{(k)}$  is the corresponding eigenvector.

Jacobi method only 1st iteration calculate kerni thi 2 marks

**Check the Topic "Each Topic with One Solved Question." On page number 7**

Jacobi method 2 iteration calculation 5 marks

**Check the Topic "Each Topic with One Solved Question." On page number 7**

Newton Backward difference method 5 marks

**Check the Topic "Each Topic with One Solved Question." On page number 7**

Backward difference table construct 2 marks

**Check the Topic "Each Topic with One Solved Question." On page number 7**

Backward difference table construct 3 marks

**Check the Topic "Each Topic with One Solved Question." On page number 7**

Mcqs mix hi thy

**Papers Number 02**

2 Questions of 2 marks

1 question was to find the value of p from the given table using Interpolation Newton's forward Difference Formula.

**Check the Topic "Each Topic with One Solved Question." On page number 7**

2) we have to calculate the values of x, y and z for the given system of equations using Gauss Seide iterative method taking initial solution vector as (0, 0, 0)t?

**Check the Topic "Each Topic with One Solved Question." On page number 7**

Both of the questions were to find the value of theta for the given matrices using Jacobi's method ? 2x3 marks

**Check the Topic "Each Topic with One Solved Question." On page number 7**

1) Construct backward difference table for the given values ...5 marks

**Check the Topic "Each Topic with One Solved Question." On page number 7**

2) Find the solution of following system of Equations using Jacobi's iterative method upto three decimal places ? 5 marks

$$83x+11y-4z=95$$

$$7x+52y+13z=104$$

$$3x+8y+29z=71$$

taking the initial solution vector as  $(0, 0, 0)^T$

**Check the Topic "Each Topic with One Solved Question." On page number 7**

### Papers Number 03

Question: 48 (Marks: 3)

Find the value of  $\theta$  from the following matrix by jacobi's method

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

**Check the Topic "Each Topic with One Solved Question." On page number 7**

Question: 52 (Marks: 5)

Using Gauss-Jordan elimination method, solve the system of equations

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ -15 \end{bmatrix}$$

**Check the Topic "Each Topic with One Solved Question." On page number 7**

Question: 50 (Marks: 5)

Find  $y(1)$  for the following initial value problem

$$yy' = x, \quad y(0) = 2$$

Using simple Euler's method, taking  $h=1$

Regula Falsi method ka aik 5 marks ka sawal

**Check the Topic "Each Topic with One Solved Question." On page number 7**

Orthogonal matrix ki definition 3 mark ki

**Check the Topic "Each Topic with One Solved Question." On page number 7**

Crute's method ko briefly define kerna tha 3 marks

**Check the Topic "Each Topic with One Solved Question." On page number 7**

Convert the following matrix into diagonal form

$$\begin{bmatrix} 2 & 0 & 6 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

### Papers Number 04\_My Paper Spring 2011 by Ishfaq

Use Gause Sediell Method to find the solution of linear system of equations, perform only two iteration correct upto 3 decimal places. 5 marks

Very easy question tha .Just remember the process.

Prove the relation  $u=1/4(\delta^2+4)$  5 marks.

See the average function.

What is the formula for Newton's backward difference interpolation...3 marks

Answered in the file

A Matrix was given, which was Symmetric .Find the value of  $\theta$  using Jacobi's method..... 3 marks

Answered in the file

What is the formula to find value of  $\theta$  ..... 2 marks

Answered in the file

Define Extrapolation? ..... 2 mark.

### Solved Mid-Term Past Papers

#### Short Question (Set-1)

my paper was power method 5 marks ,  
from vector 3 marks,

Newton\_Repshon's forwrd distnce formula 5 marks.,

define extra polation 2 marks ,, and most of MCQ's from ,Newton\_Repshon's method

#### Short Question (Set-2)

Approximate the integral  $\int_1^{1.5} \frac{2x}{x^2+1} dx$  using Simpson's 1/3 rule and calculate the percentage error. (Take result up to 4 decimal places)

**Note: In order to get full marks do all necessary steps.**

Construct a forward difference table for the following values

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
y	0.003	0.067	0.148	0.248	0.37	0.518	0.697

**Note : In order to get full marks do all necessary steps.**

Solve the system

$$4x+3y=24$$

$$3x+4y-z=30$$

$$-y+4z=-24$$

by Gauss Seidal Method, taking  $(0, 0, 0)^t$  as initial approximation(Two iterations only and

take result up to 4 decimal places )

**Note : In order to get full marks do all necessary steps**

Let  $f(x) = x + \frac{2}{x}$ , use cubic Lagrange interpolation based on the nodes  $x_0 = 0.5, x_1 = 1, x_2 = 2$  and  $x_3 = 2.5$  to approximate  $f(1.5)$  and  $f(1.3)$ .

**Note : In order to get full marks do all necessary steps**

Approximate the Dominant Eigenvalue and corresponding Eigenvector for the matrix

$$\begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix}$$

by using Power Method. Start with  $X_0 = (1, 1, 1)^T$ . (Five iterations only and take result up to 4 decimal places)

**Note : In order to get full marks do all necessary steps**

=====>  
**Short Question (Set-4)**

=====>  
**Short Question (Set-5)**

Question No: 7 (Marks: 10)

Interpolate the value of 0.25 using Newton's forward difference formula.

x	0.2	0.3	0.4	0.5	0.6
F(x)	0.2304	0.2788	0.3222	0.3617	0.3979